MECHANICS

A TEXT-BOOK FOR ENGINEERS

BY

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PREFACE

This book is intended to give a working knowledge of the principles of Mechanics and to supply a foundation upon which intelligent study of Strength of Materials, Stresses in Structures, Machine Design, and other courses of more technical nature may rest.

In the development of this subject, emphasis is put upon the physical character of the ideas involved, while mathematics is employed as a convenient tool for the determination and expression of quantitative relations. Analytical and graphical methods are given together and each is interpreted in terms of the other. While the principal stress is placed upon Mechanics as a science, considerable attention is given to Mechanics as an art. In the text, in some of the problems, and in many of the illustrative examples, methods of calculation are suggested by means of which accurate results may be most readily obtained.

The definitions of work and potential energy, together with the solution of problems of statics by the method of virtual work are given early. In the treatment of dynamics, the definitions of kinetic energy and its application to the conditions of variable motion are introduced as soon as possible.

In equations involving acceleration or energy, the common commercial units are employed—the pound mass as the unit of mass and the weight of the pound mass as the unit of force. In order to clear up the confusion which results from the fact that physicists use one set of units while some engineering writers use another, Chapter XVIII is devoted to a discussion of the various systems.

The author acknowledges his obligations to many of his colleagues who have assisted in the preparation of this book. P. W. Ott of the Department of Mechanics checked the problems of several chapters. S. A. Harbarger of the Department of English read all the manuscript and assisted in the final revision. Professor O. E. Williams of the Department of Engineering

Drawing supervised the preparation of the drawings, and Professor Robert Meiklejohn of the same department made many valuable suggestions.

J. E. Boyd.

Columbus, Ohio,
May 13, 1921.

CONTENTS

NOTATION

Symbols frequently used in this book are:

a = linear acceleration; apparent moment arm; length of balance beam; radius of circle; distance on figure.
$\mathbf{a}$ = a vector of length a.
A = Area; a force (in few cases).
b = breadth; base of triangle; a distance.
$\mathbf{b}$ = a vector of length b.
B = a force (in few cases).
c = a distance; a constant of the catenary; distance of center of gravity of balance beam below central knife-edge.
$\mathbf{c}$ = a vector of length c.
C = integration constant; a force (in few cases).
d = diameter; a distance; pitch of screw; distance of central knife-edge above end knife-edges; distance between parallel axes.
$\mathbf{d}$ = a vector of length d.
e = base of natural logarithms; a distance.
E = electromotive force; modulus of elasticity.
f = coefficient of friction.
F = force; total force of friction.
F_w = weight.
g = acceleration of gravity; a constant, 32.174.
h = height; height of triangle.
hp = horsepower.
H = product of inertia.
H_0 = product of inertia for axes through center of gravity.
$\mathbf{H}$, H, H_x, H_z = horizontal force; horizontal vector.
I = moment of inertia; electric current.
I_0 = moment of inertia for axis through the center of gravity.
I_x = moment of inertia with respect to the X axis.
I_y = moment of inertia with respect to the Y axis.
I_{max}, I_{min} = maximum and minimum moments of inertia.
J = product of inertia.
k = radius of gyration; a constant.
k_0 = radius of gyration for axis through the center of gravity.
K = force which deforms a spring 1 foot; a constant; integration constant; coefficient of discharge.
l = length; length of simple pendulum.
m = mass in pounds, grams, or kilograms.
$m.e.p.$ = mean effective pressure.
M = moment.

N = force normal to surface.
p = a small weight.
P, P = a force; force which causes an acceleration.
Q = quantity of liquid.
Q, Q = a force.
r = radius; amplitude of vibration.
$\bar{r}$ = radius to center of gravity.
R = resistance in ohms.
R, R = resultant force; reaction of support.
s = length; length of catenary from lowest point.
S, S = equilibrant; a force; unit stress.
t = time; time of vibration; thickness.
t_c = time of a complete period.
T = torque.
T, T = tension; tension in a cord.
U = kinetic energy; work.
v = linear velocity.
V = volume.
V, V = a vertical force.
w = weight per unit length.
w' = weight per unit of horizontal distance.
W = weight.
x, y, z = distances.
$\bar{x}, \bar{y}, \bar{z}$ = coördinates of center of gravity.
X, Y, Z = coördinate axes.
y_c = distance to center of pressure.
α = angular acceleration; any angle; angle with X axis; angle of contact with belt.
β = angle with Y axis; any angle.
γ = angle with z axis; any angle.
dx, dy, dz = small increments of x, y, and z.
ρ = density.
ϕ = an angle; angle of friction.
θ = an angle; angular displacement.
Σ = a summation.
ω = angular velocity.

MECHANICS

CHAPTER I

FUNDAMENTAL IDEAS

1. Mechanics.—Mechanics is the science which treats of the effect of forces upon the form or motion of bodies. The science of mechanics is divided into *statics* and *dynamics*. When the forces which act on a body are so balanced as to cause no change in its motion, the problem of finding the relations of these forces falls under the division of *statics*. When the forces which act on a body cause some change in its motion, the problem of finding the relation of the forces to the mass of the body and to the change of its motion falls under the division of *dynamics*. The division of dynamics is frequently called *kinetics*.

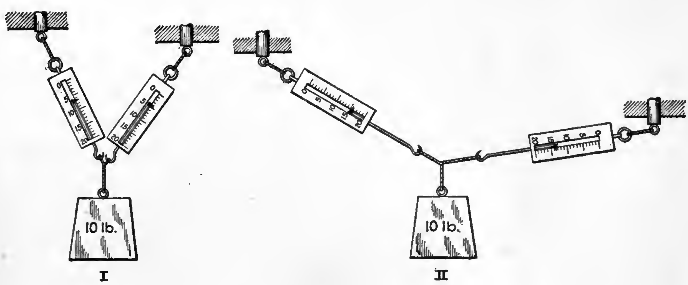


FIG. 1.

2. Illustrations.—Fig. 1 is an example of a problem of statics. The figure shows a 10-pound mass which is supported by two spring balances. In Fig. 1, I, the balances are nearly vertical. One balance reads a little less than 5 pounds and the other balance reads over 6 pounds. In Fig. 1, II, each balance makes a large angle with the vertical. The right balance, which is nearly horizontal, reads 13 pounds and the left balance reads 15 pounds. In this position, the reading of each balance is greater than the entire weight which is supported by the two balances. In each position, the balance which is the more nearly vertical gives the larger reading. The problem of finding the relation between the pulls which these balances exert and the angles which they make with the vertical is a problem of *statics.*

In Fig. 2, the mass of 10 pounds is supported by one spring balance and by a cord which runs over a pulley and carries a mass of 8 pounds on its free end. This system will come to rest in a definite position. If moved from this position, the system will return to it after a few vibrations. The problem of finding this position and the tension in the spring balance is a problem of *statics.*

If the cord which runs over the pulley of Fig. 2 is cut or broken, the 10-pound mass will swing back and forth as a pendulum and will finally come to rest with the balance in a vertical position.

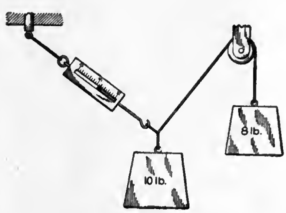


FIG. 2.

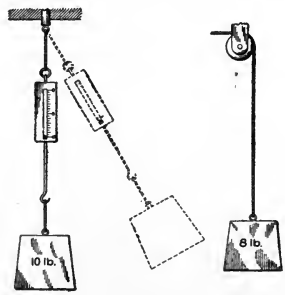


FIG. 3.

This final position is shown in Fig. 3. The 8-pound mass will fall vertically downward with increasing speed. As the 10-pound mass swings back and forth, the pull on the balance will change. The problem of finding the position of either body at any time after the cord has been severed, and the problem of finding the pull of the spring balance are problems of *dynamics.*

3. Fundamental Quantities.—The problem of Fig. 2 involves position and direction. These are properties of *space.* The problem also involves the pull of the cord and of the spring balance. These are forces. Every problem of statics involves the two fundamental ideas of *space* and *force.* If a body of different mass were used in place of the 10 pounds, the force and space relations would be changed. In this problem, the forces depend upon the masses. All problems of statics involve *force* and *space.* Most problems of statics involve also *mass.*

A problem of dynamics differs from one of statics in the fact that it involves the element of *time.*

The four fundamental quantities of mechanics are *space, mass, force,* and *time.* A problem of dynamics includes all four of these quantities. A problem of statics may include all except time.

Space, mass (or matter), force, and time are elementary. None of them may be reduced to anything more simple. Consequently, it is useless to attempt to define them. On the other hand, everyone has a clear knowledge of these quantities. This knowledge has been gained through one or more of his senses as a part of the experience of his lifetime.

4. Standards and Units.—While space, mass, time, and force can not be defined, they may all be measured. To measure any quantity, it is necessary to have a *unit* of measure. If measurements are to be taken at diverse times and places, it is necessary to have some *standard unit* to which all other units are referred. These units of measure were originally arbitrarily chosen. Other units might have been selected just as well. When a particular unit has once been adopted, however, it is important that its value be preserved without change, in order that physical measurements separated by wide intervals of time may be accurately compared.

5. Length.—Space in one direction is *length.* There are two official standards of length preserved by the Bureau of Standards at Washington. These are the Standard Yard, which is practically equal to the British Imperial Standard Yard, and the Standard Meter, which is a copy of the International Standard.

The length of the International Meter in terms of the wave length of cadmium vapor light has been carefully determined by Michelson. If this standard bar and the copies preserved by various nations should undergo any change, the magnitude of the variation may be found by a new comparison with the wave length of this light.

The *foot* is the *unit* of length which is commonly used by American engineers. A foot is one-third the length of the Standard Yard. Physicists use the centimeter as the unit. In countries where the metric system has been adopted, engineers employ the meter as the unit of length.

Length measurements are generally made by means of the sense of sight. Sometimes the sense of touch, the sense of hearing, or the muscular sense is used in comparing two lengths. The vision, however, is employed to get the actual reading.

Space in one dimension is *length;* space in two dimensions is area; space in three dimensions is volume.

The idea of space, including length and direction, is gained by the child through the sense of touch, the muscular sense, and the sense of sight. The sense of hearing also assists in determining direction.

The relation between the metric system and the inch is given with sufficient accuracy for most purposes by

$$39.370 \text{ inches} = 1 \text{ meter.}$$
$$2.540 \text{ centimeters} = 1 \text{ inch.}$$

Problems

1. Calculate the length of a foot in centimeters and memorize the result.

2. Calculate the length of a meter in feet and memorize the result.

3. Find the length of a kilometer in feet and in miles and compare the results with some reference book.

4. Express 100 meters in yards and 440 yards in meters.

5. By logarithms find the number of square inches and square feet in one square meter.

6. A hectare is 100 meters square. Find the value of a hectare in acres.

7. Using five-place logarithms, find the number of cubic centimeters in one cubic inch. Compare with some handbook.

8. A liter is a cubic decimeter. Find the relation between the liter and the U. S. liquid quart.

6. Time.—The *standard* of time measurement is the *mean solar day.* The unit commonly employed in problems of mechanics is the *mean solar second.* The subdivision of the solar day into hours, minutes, and seconds is made by means of the vibration of pendulums or other mechanical devices. In making time measurements, the senses of sight and hearing are used in connection with these mechanical timepieces.

The child gains his ideas of time from the succession of events as revealed through any or all of his senses.

7. Matter and Mass.—From the mechanical standpoint, at least, time and space are simple and easily measured. Time possesses a single property, that of extent; space has extension or length in three dimensions. Matter, on the other hand, possesses many properties, some one of which must be selected and defined as measuring the amount of material in a given body. Volume might be chosen as the measure of the quantity of material. It is found, however, that the amount of a given material in a given volume may be greatly changed by pressure. It is also found that equal volumes of different materials differ greatly in their mechanical effects. It is evident, then, that some other property must be selected to designate the amount of matter.

In Fig. 4, *A* represents a block of soft rubber resting on some convenient support. In Fig. 4, II, a body *B* has been placed on this block of rubber. The length of the block is found to have been shortened. If *B* is removed, the block *A* returns to its original length. If a body *C* is now placed on *A*, there is again a change in length. If the change in length of *A* due to the body *C* is the same as that due to the body *B*, the bodies *B* and *C* are said to contain equal amounts of material. The amount of material (or matter) in a body, as thus defined is called the *mass* of the body. Two bodies have equal masses if they produce equal deformations in a third body when they are applied to it in exactly the same way. The ordinary spring balance is a common form of a third body for the comparison of masses.

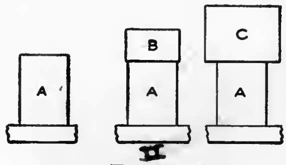


FIG. 4.

Instead of being supported on an elastic body, the bodies *B* and *C* may be carried on the hand or shoulder of the observer. The deformation of his muscles is accompanied by a sensation, called the muscular sense, which enables him to judge roughly which body has the greater mass.

A second method of measurement of mass is by means of *inertia*. This involves the conditions of change of motion and will be considered in Chapter XVII.

The child gains an idea of mass in the mechanical way by means of the muscular sense and the sense of touch as experienced when he supports bodies free from the earth, stops them when moving, or otherwise changes their motion. The concept of matter in general is gained through all the senses.

The pound is the common unit of mass. In the metric system the unit is the kilogram. Physicists and chemists use chiefly the gram. Units of mass are generally called "weights." A so-called 10-pound weight as used on a beam balance is a 10-pound mass.

To convert from the metric system to the avoirdupois system, the relations are,

$$15432 \text{ grains} = 1 \text{ gram},$$
$$453.6 \text{ grams} = 1 \text{ pound}.$$

The official Standard Pound and Kilogram are preserved by the Bureau of Standards:

Problems

1. Find the number of pounds in 1 kg. correct to four significant figures.

2. The mass of one cubic centimeter of water at 4°C. is practically one gram. Find the mass of one cubic foot of water in pounds.

8. Force.—In Fig. 4, the rubber block *A* is shortened when the body *B* is placed on it. The body *B* is said to exert a force on *A* at their surface of contact. The block *A* is also said to exert a force on *B* in the opposite direction. Force is something which may exist between two bodies or between two parts of the same body. Forces occur in pairs. There is a force from the first body to the second body, and an equal and opposite force from the second body to the first body. Newton stated this fact in what is called the Third Law of Motion: "Action and reaction are equal and opposed to each other."

A force always causes some change in the *dimensions* of a body. A force always *tends* to produce some change in the *motion* of the body upon which it acts, and does cause some change unless it is balanced by an equal and opposite force acting on the body, or by a number of forces equivalent to an equal and opposite force.

Force is recognized and measured by means of the change in the dimensions and form of elastic bodies, by the muscular sense, and by the change in the motion of bodies of known mass.

9. Weight.—When the masses of two bodies are compared by means of the muscular sensation experienced in lifting them, the observer really gets a comparison of two *forces*. The comparison of mass is indirect. If the observer were at the center of the Earth, there would be no muscular sensation so long as there were no change in the motion of the body. The Earth exerts a force on all bodies. This force is in the form of a pull directed toward the center of the Earth. This pull or *attraction* is called the *weight* of the body. When a body is supported and thus prevented from moving toward the earth, the support exerts a force upward which is equal to the weight of the body.

When a *physicist* speaks of the *weight* of a body, he always means the *force* with which the Earth attracts the body. In the common use of the word, *weight* generally means *mass*. When one says that the weight of a bar of iron is 16 pounds, he is usually thinking of the amount of iron and not of the force required to lift it. Much confusion has resulted from the failure to designate clearly which of these two meanings is intended.

10. Relation of Mass to Weight.—The definition of mass in Art. 7 may now be extended. *Two bodies have equal masses if, at a given point, they are attracted toward the Earth with equal force.* The determination of mass by means of a spring balance or a beam balance is accomplished indirectly by a comparison of forces, with the tacit assumption that equal forces produce equal effects. The first definition of mass is: *The mass of a body is proportional to its weight.* If F_w is the weight of the body in some convenient unit, and m is its mass, the definition may be expressed algebraically by the equation,

$$F_w = km, \tag{1}$$

in which k is a constant. The numerical value of k depends upon the units used in expressing F_w and m. These units may be so chosen that k is unity. If m is expressed in pounds of mass and F_w is in pounds of force, then $k = 1$, and

$$F_w = m. \tag{2}$$

Equation (2) states that the mass of a body in pounds is equal to its weight in pounds.[1] The word *pound* has two meanings in mechanics. It may be used to designate the amount of material (mass) or to express the force of attraction toward the Earth (weight as meant by the physicist). In a similar way, the weight of one kilogram of matter is one kilogram, and the weight of one gram of matter is one gram. With the systems of units in everyday use, k is *unity*. In some systems, k is not *unity*. In the absolute system of units $k = g$, and $F_w = mg$. The weight of a mass of m grams in that system is mg dynes.

The absolute systems of units are not used by engineers in the solution of problems of statics. In all such problems, Equation (2) applies. The weight of a body is numerically equal to its mass. The absolute systems of units and a second

[1] A formula is merely a brief statement of the relation of quantities. The letters of a formula represent the *number* of units which express the magnitude of the quantity. In the above equations, m represents the *number* of pounds, the *number* of *kilograms*, the *number* of tons, or the *number* of grams of material in the body under consideration. Similarly F_w represents the *number* of units of force in pounds, kilograms, tons, grams, poundals, or dynes.

definition of mass will be considered in this book in Chapters XVII and XVIII.

11. Variation of Weight with Latitude and Altitude.—The weight of a body at any given position varies as its mass. It has been shown by experiment that the weight of a body varies inversely as the square of its distance from the center of the Earth. If the Earth were a sphere and did not rotate on its axis, the weight of a body would be the same at all points at the same level on its surface. Since the Earth is a sphoroid with its polar radius about 13 miles shorter than its equatorial radius, the weight of a body increases with latitude. While a pound mass is an invariable quantity, the *weight* of a pound mass, as measured by a *spring balance,* varies with the latitude. If two masses have equal weights at one locality, their weights will be equal at any other locality. Masses may be compared by weighing at any point. A spring balance, however, which has been calibrated by means of a standard weight at one locality, can not be used for the accurate determination of mass at another locality. (No one would do so on account of the variation of the spring, even if the force of gravity were constant.)

Since the practical *units* of force are determined from the *weights* of the standard units of mass, it is necessary to choose some standard location for the definition of these units of force. The sea level at 45° latitude is taken as this standard location.

A pound force is defined as the weight of a pound mass at the standard location. A pound mass will weigh 0.997 lb. at the Equator and 1.003 lb. at the Poles on a spring balance which is correct at the standard latitude. This difference, while important in the determination of physical constants, is usually neglected in engineering calculations.

Problems

1. Taking the equatorial diameter of the Earth as 8000 miles and the polar diameter as 26 miles less, and neglecting the effect of the rotation of the earth, what is the weight at the Pole of a body which weighs 1 pound at the Equator, if both weighings are made on the same spring balance?

2. What is the relative change in the weight of a body when it is taken from a point at the sea level to a point one mile higher?

CHAPTER II

QUANTITY AND CALCULATIONS

12. Representation of Quantity by Numbers.—There are several ways of representing the magnitude of a quantity. The most common method is by means of numbers, as 6 feet, 8 pounds, 10 seconds. A number expresses the magnitude of the quantity in terms of the unit and means little to one who does not possess a definite idea of the magnitude of the unit. Two such numbers give a clear notion of the *relative* size of quantities without conveying any information as to the *actual* size of either. Any-one will know that 20 dekameters is twice 10 dekameters, without having any idea as to the size or nature of a dekameter. If he has learned that a dekameter is 10 meters and that a meter is 3.28 feet, he will calculate that one dekameter is approximately 2 rods and that 20 dekameters is nearly 40 rods, or he may reduce to yards and think of 10 dekameters as a little over 100 yards. A Frenchman, on the other hand, who thinks in the metric system, must translate rods and yards into meters before he can have a real idea of their meaning.

13. Representation of Quantity by Lines.—The *relative* magnitudes of several quantities are frequently represented to the eye by means of straight lines as in Fig. 5. Economic data, such as the population and area of countries and cities, the production and consumption of commodities, etc., are commonly shown in this way. These lines may be horizontal, as in Fig. 5, or vertical with their lower ends on the same horizontal line.

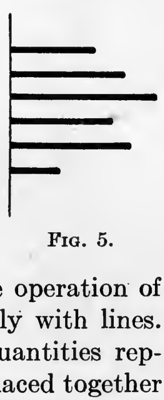


FIG. 5.

For most purposes such lines are merely used to *express* magnitude to the eye. The necessary calculations are made by means of numbers. The operation of addition, however, may be performed conveniently with lines. In Fig. 6, it is desired to find the sum of the quantities represented by the lines *ab* and *cd*. The lines are placed together so as to form one continuous line without overlapping. The total line thus formed is the sum of the lines. As may be seen from Fig. 6, it is immaterial in what order the lines are placed together.

For subtraction, especially when the remainder is negative, it is desirable to adopt some convention as to positive and negative direction. Horizontal lines extending toward the right and vertical lines extending upward are regarded as positive. In Fig. 6, the line *ab* runs from *a* to *b*. The left end, *a*, is called the *origin*, and the right end, *b*, is called the *terminus*. To find *ab* + *cd*, the origin of the second line is placed at the terminus of

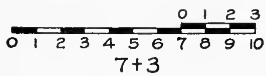


FIG. 7.

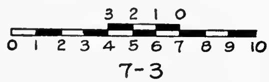


FIG. 8.

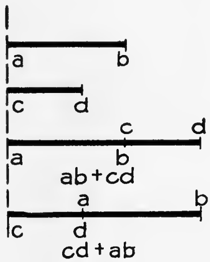


FIG. 6.

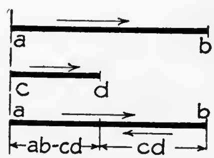


FIG. 9.

the first line. The sum of the two lines extends from the origin of the first line to the terminus of the second line. This corresponds with ordinary addition of numbers. To get the sum of 7 + 3 begin at 7, which is the terminus of the first number, and count forward 3 steps. This is shown graphically by Fig. 7.

Subtraction is the addition of a negative quantity. To get 7–3, begin at 7 and count backward 3 steps. This is shown by Fig. 8. To get *ab* – *cd*, begin at the terminus of *ab* and measure the length of *cd* toward the left. This is shown in Fig. 9. The arrows in Fig. 9 give the direction of the motion.

14. Vectors.—A quantity which has both magnitude and direction is called a vector quantity. Force is an example of this kind of quantity. In Fig. 10, *ab*, *cd*, and *ef* are vectors in the plane of the paper. The vectors *ab* and *ef* are equal, since they have the same direction and equal length.

In the vector *ab*, the point *a* is the *origin* and the point *b* is the terminus. The vector is considered as extending from the origin to the terminus, as is indicated by the arrow. The arrow points *from* the origin and *toward* the terminus. The direction of the arrow is the positive direction of the vector.

When a vector is represented by a single letter, that letter is usually printed in **black face type.** In Fig. 11, **a** and **b** represent two such vectors. The arrow shows the origin, terminus, and direction.

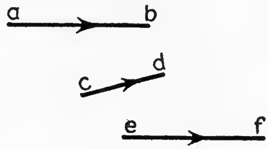


FIG. 10.

A vector is described in words by giving its direction and its length. When the plane of the vector is known, a single angle is sufficient to designate its direction. For instance, a given vector is 8 feet in length and makes an angle of 35 degrees with the horizontal. When the vectors under consideration are not all in the same plane, two angles are required to express the direction of each vector. For instance, a vector is 8 feet in length and makes an angle of 40 degrees with the vertical in a vertical plane which is north 25 degrees east.

When it is desirable to distinguish a quantity which has magnitude but not direction from a vector, such a quantity is called a *scalar* quantity. The mass of a body or the number of individuals in a group is a scalar quantity. In an algebraic formula in which only the magnitude of a vector is represented by a letter while its direction is expressed in terms of angles, the letter is printed in *Italics* instead of in **black face type.** In this book, a letter (such as **P** or **Q**) is used to represent a force, which is a vector. When emphasis is put on both the direction and magnitude of the force, the letter is printed in **black face type.** When only the magnitude is stressed, it is printed in *Italics*.

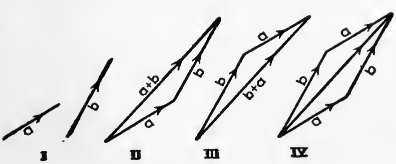


FIG. 11.

15. Addition of Vectors.—The addition of vectors is defined in the same way as the addition of lines (Art 13). The origin of the second vector is placed at the terminus of the first vector. The line which extends from the origin of the first vector to the terminus of the second vector is the sum of the two vectors.

Fig. 11, II, shows the addition of the vectors **a** and **b** to get their sum **a** + **b**. Fig, 11, III, shows **b** + **a**. The vector **a** is first in Fig. 11, II, and the vector **b** is first in Fig. 11, III. Fig. 11, IV, shows both additions in one diagram. The additions begin at a common origin *O*. Since the two lines **a** are equal and parallel and the two lines **b** are also equal and parallel, the four lines form a parallelogram. The diagonal of this parallelogram is the vector sum and it is immaterial in what order the two vectors are added. The sum of three vectors is found in the same way. Fig. 12, II, represents the sum of three vectors. Fig. 12, I, may be regarded as a graphical statement of the vectors which are to be added. In this figure all the vectors start from a common origin. In Fig. 11, I, on the other hand, the vectors **a** and **b** start from different origins.

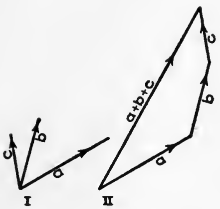


FIG. 12.

There are two methods of finding the vector sum. These are the *graphical* method in which the lengths and angles are *measured*, and the *trigonometric* (or algebraic) method in which the lengths and angles are *computed*.

Problems

1. Given two vectors, **a** = 15 ft. at 0 degrees, **b** = 12 ft. at 40 degrees. Solve graphically for the vector sum, **a** + **b**. Use the scale of 1 inch = 5 feet. Measure the vector sum and express the result in feet. Measure the angle of the vector sum with the first vector and express the result in degrees.

First construct the statement to scale as shown in Fig. 13, I. Then draw **a** in Fig. 13, II, equal and parallel to **a** of the statement. From the terminus of **a** draw **b** equal and parallel to **b** of the statement.

2. Solve problem 1 for the vector sum **b** + **a**.

3. Find the sum of three vectors: 20 ft. at 0 degrees, 15 ft. at 45 degrees, and 10 ft. at 110 degrees. Use the same scale as in Problem 1.

4. Find the vector sum of 16 ft. at 10 degrees and 20 ft. at 70 degrees. Construct the 10-degree angle by means of its tangent. Construct the 60-degree angle by means of its chord. Measure the angle of the vector sum by means of its chord and check by means of the sine of the angle which the vector sum makes with a line at 90 degrees.

16. Components of a Vector.—The sum of two or more vectors is frequently called the *resultant* vector and the vectors which are added are called *components* of the resultant. The process of finding the resultant is called *composition* of vectors. The process of finding the components is called *resolution*. In Fig. 13, the vector **a** + **b** is the resultant of vectors **a** and **b**, and the vectors **a** and **b** are components of **a** + **b**.

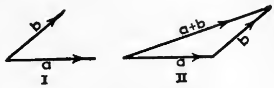


FIG. 13.

Problems

1. A vector of 25 ft. at 30 degrees is made up of two components. One of these components makes an angle of 5 degrees with the reference line and the other makes an angle of 45 degrees with the reference line. Find these components graphically.

2. A vector of 20 ft. at 45 degrees is made up of a vector **a** at 20 degrees and a vector **b** which is 12 ft. in length. Find the magnitude of **a** and the direction of **b**.

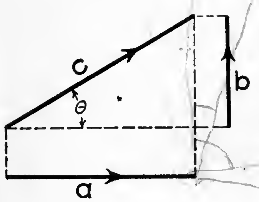


FIG. 14.

The most important kind of resolution of vectors is that in which each component is formed by the orthographic projection of the resultant upon a line along the desired direction. In Fig. 14, **c** is a vector which makes an angle θ with the horizontal. Its horizontal component is **a** and its vertical component is **b**. The lengths of these components are given by the equations

$$a = c \cos \theta,$$
$$b = c \sin \theta.$$

(In these equations, the letters a, b, and c represent magnitudes only. For that reason they are printed in *Italics* instead of in **black face type.**)

Figure 15 shows two such orthographic components, which with their resultant form a right triangle.

When the term *component* is used without qualification, the orthographic component is generally meant. The process of finding the orthographic component of a vector in a given direction is called *resolution* in that direction.

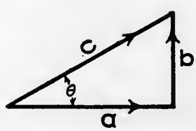


FIG. 15.

Problems

Solve the following problems graphically and trigonometrically.

3. A vector of 60 ft. is north 27 degrees east. Find its component north and its component east. *Ans.* 53.46 ft. north; 27.24 ft. east.

4. A vector of 45.67 ft. is directed north 27° 07′ west. Find its component north and its component west by means of logarithmic functions.

5. Find the component of the vector of Problem 3 along a line which is north 45 degrees east. Check by means of the sum of the components along this direction of the east and north components as given in the answer to Problem 3.

6. The horizontal component of a vector is 18.24 ft. and the vertical component is 12.48 ft. By means of the tangent find the angle which the vector makes with the horizontal. Then find the length of the resultant vector by means of the cosine (or secant) of this angle. Check by projecting the horizontal and vertical components upon the line of the resultant and adding the components thus found (Fig. 16).

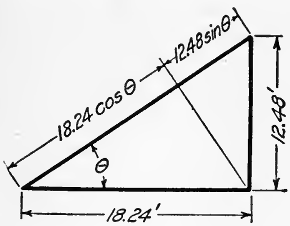


Fig. 16.

7. The horizontal component of a vector is 27.734 ft. The vertical component is 18.245 ft. Find the direction of the resultant vector by means of the logarithmic tangent. Find the magnitude of the resultant by means of the logarithmic cosine. Check by projections as in Problem 16.

17. Computation of the Vector Sum.—Problem 6 of the preceding article is an example of the *computation* of the vector sum when there are only two vectors to be added, and these vectors are at right angles to each other. In Fig. 13, only two vectors are to be added, but these are not at right angles to each other. In this problem, the vector sum may be found by the formulas for oblique-angled triangles. The unknown side may be computed by the *Law of Cosines* and the angles then found by the *Law of Sines.* Another method is to find first the unknown angles by means of the formula which expresses the relation of the tangents of half the sum and half the difference of two angles to the sum and difference of the sides opposite, and then to find the unknown side by the *Law of Sines.* This last method permits the use of logarithmic functions. Since neither of these methods is convenient to apply when there are more than two vectors to be added, it is best to learn a general method which is valid for all cases of vectors in a single plane.

First, find the component of each vector in the direction of some line in their plane. The algebraic sum of these components is the component of the vector sum in this direction. Then find the component of each vector in the direction of a second line which is at right angles to the first direction. The algebraic sum of these components is the component of the vector sum in this direction. These two algebraic sums represent the legs of a right-angled triangle. The hypotenuse of this triangle is the vector sum required.

Figure 17 shows the addition of three vectors **a**, **b**, and **c** in the same plane. These vectors make angles of α, β, and γ, respectively with the horizontal line. If H is the algebraic sum of the components of these vectors parallel to the horizontal line,

$$H = a \cos \alpha + b \cos \beta + c \cos \gamma.$$

If V is the algebraic sum of all the vertical components,

$$V = a \sin \alpha + b \sin \beta + c \sin \gamma.$$

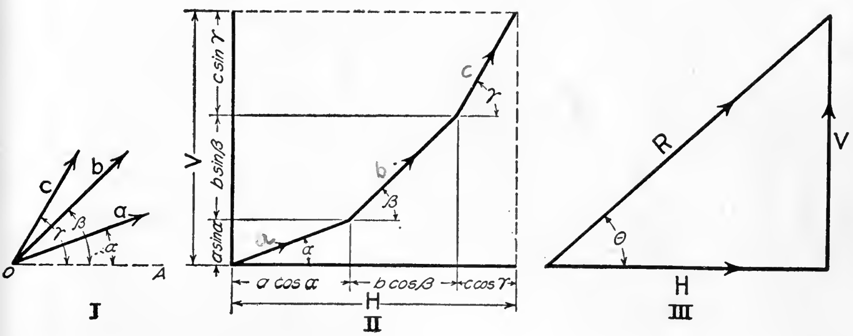


FIG. 17.

The resultant vector is the vector sum of a horizontal vector of length H and a vertical vector of length V. (Fig. 17, III.)

If θ is the angle which the resultant vector makes with the horizontal,

$$\tan \theta = \frac{V}{H}; \quad (1)$$

$$R = H \sec \theta = \frac{H}{\cos \theta}, \quad (2)$$

$$R = \frac{V}{\sin \theta}. \quad (3)$$

If H is greater than V, use Equation (2) to find R; if V is greater than H, use Equation (3). The most accurate result is obtained by this method.

It is not necessary that the direction of H should be horizontal. The resolution may be taken along any direction, and the resolution for V at right angles to the direction of H.

Example

Find the direction and magnitude of the vector sum of 12 units at 18 degrees and 15 units at 50 degrees.

The equation for the components at 0 degrees is,

$$H = 12 \cos 18° + 15 \cos 50°,$$
$$12 \times 0.9511 = 11.4132$$
$$15 \times 0.6428 = 9.6420$$
$$H = 21.0552$$

The equation for the components at 90 degrees is,

$$V = 12 \sin 18° + 15 \sin 50°,$$
$$12 \times 0.3090 = 3.7080$$
$$15 \times 0.7660 = 11.4900$$
$$V = 15.1980$$
$$\text{Tan } \theta = \frac{15.1980}{21.0552} = 0.7218,$$
$$\theta = 35° 49'.$$
$$R = \frac{21.0552}{\cos \theta} = \frac{21.0552}{0.8108} = 25.96.$$

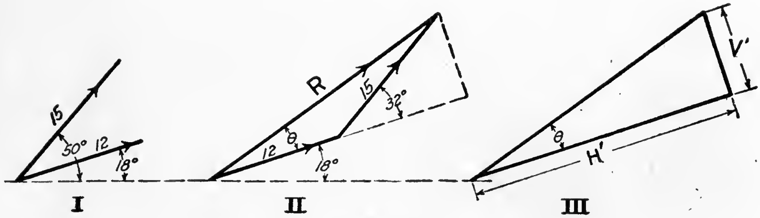


FIG. 18.

Instead of taking resolutions along the lines at 0 degrees and 90 degrees, resolve along the direction of the first vector and perpendicular to that direction (Fig. 18). If the component of the vector sum along the first direction is H' and the component at right angles to this direction is V',

$$H' = 12 + 15 \cos 32°,$$
$$H' = 12 + 15 \times 0.8480 = 12 + 12.7200 = 24.7200,$$
$$V' = 15 \sin 32° = 15 \times 0.5299 = 7.9485.$$
$$\text{Tan } \theta = \frac{7.9485}{24.7200} = 0.3215,$$
$$\theta = 17° 49'.$$
$$R = \frac{24.7200}{\cos \theta} = \frac{24.7200}{0.9520} = 25.96 \text{ units}.$$

This will be recognized as the method of computing oblique-angled triangles by means of right triangles.

The computation of the components in the foregoing example has been carried to four decimal places, which gives five or six significant figures in the resultant products. As only four-place trigonometric functions have been used, the last figure or the last two figures may be incorrect. The last figure of each product may well be dropped and the results written $H = 11.413 + 9.642 = 21.055$. Many computers would drop the last two figures and give $H = 21.06$. This, however, should not be done. An error of unity in the last figure of 21.06 is greater relatively than an error of unity in the fourth place of the cosines which are used in the computation. For accurate work it is a good rule to *carry the calculations one figure farther than the data,* and finally drop the last figure from the result. This method prevents errors in the calculations, which may greatly exceed the errors of the data.

In the foregoing example, it has been assumed that the length of each vector is correct to four significant figures. In many cases the degree of accuracy is expressed by the addition of zeros. A length of 12 feet, which is correct to hundredths of a foot, is written 12.00 ft. This, however, is not always done.

In these examples, it is also assumed that the angles are correct to minutes. An angle which is given as 32° will be understood to be 32° 00′.

Problems

1. Find the magnitude and direction of the resultant of 24.8 ft. at 0 degrees and 22.8 ft. at 65 degrees. *Ans.* 40.16 ft. at 30° 58′.

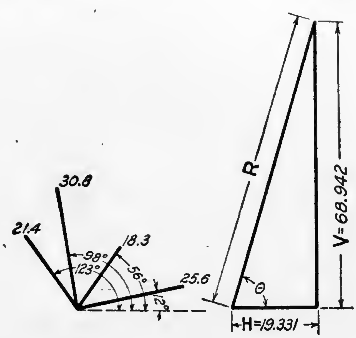


FIG. 19.

2. Find the magnitude and direction of the resultant of 24 units at 10 degrees, 32 units at 28 degrees, and 16 units at 70 degrees. Resolve along 0 degrees and 90 degrees.

3. Check Problem 2 by resolving along the line at 10 degrees and along the line at 100 degrees.

4. Find the direction and magnitude of the vector sum of 25.6 units at 12 degrees, 18.3 units at 56 degrees, 30.8 units at 98 degrees, and 21.4 units at 123 degrees.

Where there are several vectors, it is convenient to arrange the solution in tabular form.

Length	Angle	Cosine	Sine	*H* comp.	*V* comp.
25.6	12°	0.9781	0.2079	25.039	5.322
18.3	56°	0.5592	0.8290	10.233	15.171
30.8	98°	−0.1392	0.9903	− 4.287	30.501
21.4	123°	−0.5446	0.8387	−11.654	17.948
				19.331	68.942

$$\text{Cotan }\theta = \frac{19.331}{68.942} = 0.2804,$$

$$\theta = 74° \ 20'.$$

$$R = \frac{68.942}{\sin\theta} = \frac{68.942}{0.9628} = 71.60.$$

5. Solve Problem 4 by resolutions along 12 degrees and 102 degrees.

Ans. $H' = 33.243$; $V' = 63.419$; cotan $\theta = 0.5242$.

6. Solve problem 1 for the vector sum by means of the *law of cosines.* Then find the direction by the *law of sines.*

7. Find the vector sum of 24.62 units at 0 degrees and 18.28 units at 62 degrees. First, find the angles by means of the relations of the tangents of half the sum and half the difference. Then find the magnitude of the vector sum by the *law of sines.*

8. A vector of 24.6 units at 20 degrees and a vector of 16.8 units at an unknown angle have a resultant of 12.4 units. Find the unknown directions by means of the tangents of the half angles.

18. Vector Difference.—Subtraction is the addition of a negative quantity. If a negative quantity be regarded as having direction, its direction is opposite to the direction of a positive quantity. *A negative vector is opposite in direction to a positive vector.* If vector **a** = 4 feet at 20 degrees, − **a** = 4 feet at 200 degrees.

In Fig. 8, the number 3 is subtracted from 7 by counting backwards three units from the terminus of 7. Ordinary addition and subtraction may be regarded as special cases of vector addition and subtraction in which all the positive vectors are in the same direction. Subtracting 3 is equivalent to adding − 3 or counting three units in the negative direction.

Figure 20 shows two vectors, **a** and **b**. To get **a** − **b**, the vector **a** is first drawn. From the terminus of vector **a** of Fig. 20, II, the vector − **b** is drawn opposite to the direction of vector **b** of

Fig. 20, I. The vector **c** from the origin of **a** to the terminus of −**b** is the required difference.

$$\mathbf{a} - \mathbf{b} = \mathbf{c},$$
$$\mathbf{a} = \mathbf{c} + \mathbf{b}.$$

Figure 20, III, shows another method of getting the same result. If $\mathbf{c} = \mathbf{a} - \mathbf{b}$, $\mathbf{a} = \mathbf{b} + \mathbf{c}$. The vector **c** is the vector which must be added to vector **b** to get vector **a**. Vectors **a** and **b** are drawn from the same origin and the terminus of **b** is joined to the terminus of **a**. This vector **c**, with its origin at the terminus of **b** and its terminus at the terminus of **a**, is the required difference. Fig. 20, IV, shows the two methods combined. The

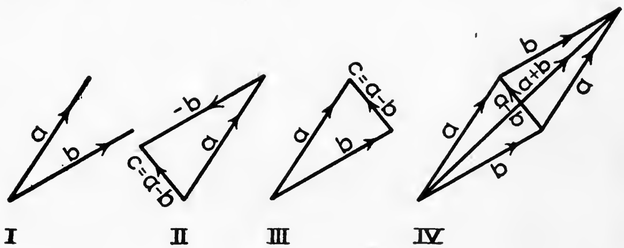


FIG. 20.

vectors **a** and **b** form the sides of a parallelogram. The diagonal which starts at the origin of the first two vectors is the vector sum. The other diagonal is the vector difference.

Problems

1. Given a vector $\mathbf{a}$ = 22 units at 0 degrees and a vector $\mathbf{b}$ = 22 units at 30 degrees. Find $\mathbf{b} - \mathbf{a}$. *Ans.* $\mathbf{b} - \mathbf{a}$ = 11.39 units at 105°.

2. A vector $\mathbf{a}$ = 17.28 ft. at 56° and a vector $\mathbf{b}$ = 27.34 ft. at 24 degrees. Find $\mathbf{a} - \mathbf{b}$. Solve graphically, then solve trigonometrically.

3. A vector $\mathbf{a}$ = 12.4 ft. at 20°, a vector $\mathbf{b}$ = 17.2 ft. at 45°, and a vector $\mathbf{c}$ = 19.2 ft. at 64°. Find $\mathbf{a} + \mathbf{b} - \mathbf{c}$. First solve graphically. Then solve by resolutions as in Art. 17. *Ans.* 15.42 ft. at 356° 49′.

4. Check Problem 3 by resolutions along 20° and 110°.

5. In Problem 3, find $\mathbf{a} - \mathbf{b} - \mathbf{c}$.

19. Vectors in Space.—A *vector* in a *plane* may be specified numerically by two quantities. These may be its length and its angle with some axis, which correspond with polar coördinates, or its components along two axes, which correspond with Cartesian coördinates.

To specify fully a *vector* in *space* requires three quantities. These may be its *length* and *two angles* or its *components* along *three axes* which are not all in one plane.

Figure 21 shows one way of expressing the angles of a vector in space. The vector of length l makes an angle β with the Y axis, while the plane which passes through the vector and the Y axis makes an angle Φ with the XY plane. This is equivalent to *co-latitude* and *longitude* on a sphere. The Y axis may be regarded as the polar axis of the sphere. The angle β is equivalent to the co-latitude and the angle Φ is equivalent to the longitude. These coördinates are called spherical coördinates.

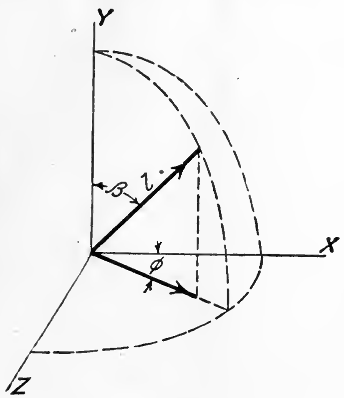


FIG. 21.

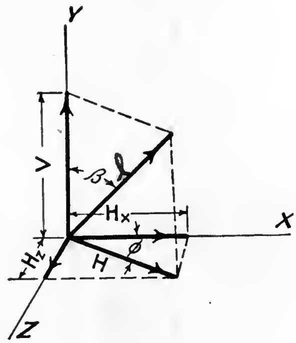


FIG. 22.

If H is the component in the XZ plane and V is the component parallel to the Y axis, Fig. 22,

$$H = l \sin \beta, \quad (1)$$
$$V = l \cos \beta. \quad (2)$$

If H_x is the component along the X axis and H_z is the component along the Z axis,

$$H_x = H \cos \Phi = l \sin \beta \cos \Phi, \quad (3)$$
$$H_z = H \sin \Phi = l \sin \beta \sin \Phi. \quad (4)$$

Problems

1. A vector, 25 feet in length, makes an angle of 35 degrees with the horizontal in a vertical plane which is south 25 degrees east. Find its vertical component and its horizontal components east and south.

Ans. V = 14.34 ft.; H_e = 8.65 ft.; H_s = 18.56 ft.

2. A vector, 64.24 feet in length, is in a vertical plane which is north 37 degrees east. The elevation of the vector is 47 degrees. Find its component north, its component east, and its component vertical.

3. The vertical component of a given vector is 25.6 feet. The horizontal component east is 16.8 feet. The horizontal component south is 14.4 feet. Find the direction and magnitude of the vector.

20. Vectors by Direction Cosines.—A second method of expressing the direction of a vector is by means of the angles which it makes with two of the coördinate axes. In Fig. 23, the vector *OP* of length l is drawn as the diagonal of a rectangular parallelopiped. Three edges of this parallelopiped lie in the axes of coördinates. The angle between the vector and the X axis is α. The angle between the vector and the Y axis is β; and the angle between the vector and the Z axis is γ. These angles are the *direction angles* of the vector, and their cosines are called the *direction cosines.*

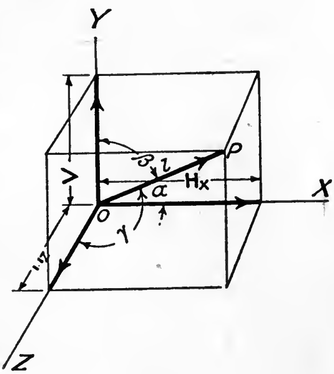


FIG. 23.

$$H_x = l \cos \alpha; \tag{1}$$
$$V = l \cos \beta; \tag{2}$$
$$H_z = l \cos \gamma. \tag{3}$$

Since $l^2 = H_x^2 + V^2 + H_z^2 = l^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{4}$$

Equation (4) is a fundamental formula of solid analytic geometry.

In the statement of a problem, two of the direction angles are given and the third is found by means of Equation (4).

It will be noted that Equation (2) is the same as Equation (2) of Art. 19. One angle is measured in the same way in both methods.

(In works of mathematical physics, the components which are here written H_x, V, and H_z are written X, Y, and Z.)

It is not necessary, of course, that these axes be horizontal and vertical. They may be taken in any convenient direction. In most cases of practice, however, it is easiest to measure the angles from a vertical line and from lines in the horizontal plane.

Example

A vector, 30 feet in length, makes an angle of 70 degrees with the horizontal axis east, and an angle of 65 degrees with the vertical axis. Find its components along each of the coördinate axis.

$$\cos^2\gamma = 1 - \cos^2 70° - \cos^2 65° = 1 - 0.1170 - 0.1786 = 0.7044;$$

$$\cos^2\gamma = 0.7044 = \frac{1 + \cos 2\gamma}{2};$$

$$\cos 2\gamma = 0.4088;$$

$$2\gamma = 65°52', \ \gamma = 32°56'.$$

The squares of the cosines of 70° and of 65° are most easily found by means of the cosines of the double angles.

Problems

1. A vector 24 feet in length makes an angle of 65 degrees with the south horizontal line and an angle of 60 degrees with the east horizontal line. The direction of the vector is above the horizontal. Find its angle with the vertical and find its components.

Ans. $\beta = 60° \ 00'$; $H_s = 16.97$ ft.; $H_e = 12$ ft.; $V = 12$ ft.

2. A vector 40 feet in length makes an angle of 67 degrees with the vertical and an angle of 23 degrees with the east horizontal axis. Find its angle with the south horizontal axis and find its components.

3. A vector 50 feet in length makes an angle of 25 degrees with the east horizontal axis and an angle of 80 degrees with the vertical. Its direction is east of south. Find its angle with the south horizontal axis and find its components. *Ans.* $\gamma = 67° \ 20'$; $H_e = 45.32$ ft.; $H_s = 19.27$ ft.; $V = 8.68$ ft.

4. The east component of a vector is 18 feet. The vertical component is 20 feet. The length of the vector is 32 feet. Find the south component and the direction angles.

21. Addition of Vectors in Space.—To compute the vector sum of *vectors in one plane*, each vector is revolved along two axes at right angles to each other. The sum of the components along one axis forms one leg of a right-angled triangle, and the sum of the components along the other axis forms the other leg of this triangle. The hypotenuse of this triangle is the vector sum or resultant of the separate vectors. (Art. 17.) Likewise, to find the vector sum of a number of vectors in space, each vector is resolved into *three components* along axes which are mutually at right angles. The sum of the components along each axis forms one edge of a rectangular parallelopiped. The diagonal of this parallelopiped is the vector sum required.

Example

Find the vector sum of a vector 20 feet in length, which is elevated 40 degrees above the horizontal in a vertical plane north 25 degrees east, and a vector 30 feet in length, which is elevated 65 degrees in a vertical plane north 55 degrees east.

Vector.	V	H_n	H_e
20	12.856	13.883	6.475
30	27.189	7.272	10.386
—	40.045	21.155	16.861

In horizontal plane

$$\tan \phi = \frac{H_e}{H_n} = \frac{16.861}{21.155};$$
$$\log \tan \phi = 9.90147; \ \phi = 38° \ 33'.$$
$$H = \frac{21.155}{\cos \phi}; \ \log H = 1.43215; \ H = 27.05 \text{ ft.}$$
$$\text{Tan } \beta = \frac{27.05}{40.045}; \ \log \tan \beta = 9.82960; \ \beta = 34° \ 02'.$$
$$R = \frac{40.045}{\cos \beta}; \ \log R = 1.68415; \ R = 48.32 \text{ ft.}$$

Problems

1. Find the vector sum of a vector 30 feet in length, which is elevated 60 degrees in a vertical plane north 35 degrees east; a vector 20 feet in length, north 65 degrees east in a horizontal plane; and a vector 25 feet in length in a north and south vertical plane at an angle of 36 degrees north of the vertical.

Ans. $V = 46.21$ ft.; $H_n = 35.43$ ft.; $H_e = 26.73$ ft.; $\phi = 37° \ 02'$; $H = 44.38$ ft.; $\beta = 43° \ 51$; $R = 64.08$ ft.

2. In Problem 1, find the angle which the resultant makes with each of the three coördinate axes.

3. Find the vector sum of the following vectors: a vector 20.66 feet in length, which is elevated 32 degrees in a vertical plane north 16 degrees east; a vector 12.84 feet in length, which is elevated 64 degrees in a vertical plane north 30 degrees east; and a vector 18.62 feet in length, which is elevated 40 degrees in a vertical plane north 45 degrees west.

Ans. $V = 34.46$ ft.; $H_n = 31.80$ feet.; $H_e = -2.44$ ft.; $R = 46.95$ ft. at an angle of 42° 47′ with the vertical in a plane north 4° 23′ west.

4. Find the vector sum of a vector 25 feet in length in the north east quadrant at an angle of 64 degrees with the vertical and at an angle of 60 degrees with the north horizontal axis; and a vector 20 feet in length at an angle of 40 degrees with the vertical in a vertical plane, which is north 36 degrees east.

22. Graphical Resolution of Vectors in Space.—Fig. 24 shows the graphical method of finding the components of a vector when the directions are given in spherical coördinates. The vector of length l makes an angle β with the vertical in a vertical plane at an angle ϕ with the XY vertical plane. The line OP, of length l, is first drawn in the V plane at an angle β with the vertical. Its projection on the Y axis gives the V component, and its projection on the X axis gives the *length* of the horizontal component H. The line OQ of length H is revolved through an angle ϕ in the horizontal plane to the position OQ'. This line OQ' gives the location and magnitude of the horizontal component. The component of OQ' along the X axis is H_x and the component along the Z axis is H_z.

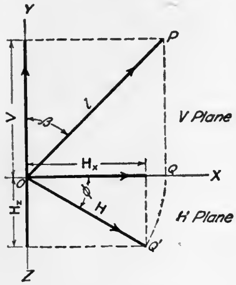


FIG. 24.

Problems

1. Solve Problem 1 of Art. 19 graphically to the scale of 1 inch = 5 feet. Also find the component in each of the three planes.

2. A vector 20 feet in length makes an angle of 35 degrees with the horizontal axis east. The vector is in a plane which makes an angle of 40 degrees with the horizontal in the octant above the horizontal plane and north of the east axis. Find the components of this vector along the east horizontal, the north horizontal, and the vertical axis.

Figure 25 shows the graphical method of finding the components of a vector when two of its direction angles are given. The

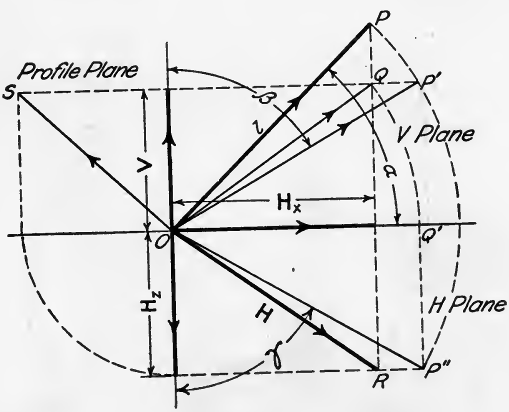


FIG. 25.

vector of length l makes an angle α with the X axis and an angle β with the Y axis. In the V plane the line OP of length l is first constructed at an angle α with the X axis. Its projection on the

X axis is the component H_x. The line OP' of length l is next drawn at an angle β with the Y axis. Its projection on the Y axis is the component V. The intersection of the horizontal line through P' with the vertical line through P gives the point Q. This point is the projection on the V plane of the terminus of the vector in space. With OQ as a radius and O as a center, an arc is described intersecting the X axis at Q'. With l as a radius and O as a center, a second arc is described in the H plane. From Q' a line is drawn in the H plane perpendicular to the X axis. This line intersects the arc of radius l at the point P''. Since OP'' is the length of the vector, and OQ' is the length of its component in the V plane, the line $Q'P''$, which completes the right triangle $OQ'P''$, is equal to the component H_z. The line OR of length H is the component in the H plane and the line OS is the component in the profile plane. It is not necessary to find these last two components, but it is sometimes desirable to have them.

Problems

3. Solve Problem 2 of Art. 20 graphically to the scale of 1 inch = 10 feet.
4. Solve Problem 3 of Art. 20 graphically to the scale of 1 inch = 10 feet.

23. Graphical Determination of the Vector Sum.—To find the sum of several vectors in space by graphical methods, each vector is first resolved into components along the three axes as in Figs. 24 and 25. To avoid confusion it is often advisable to make a separate drawing for the resolution of each vector. The components along each axis are added graphically and laid off along the corresponding axis of the final diagram. If the direction of the vector sum is desired in spherical coördinates, the construction is that of Fig. 24 in the inverse order. If the direction angles of the vector sum are required, they may be obtained by reversing the construction of Fig. 25.

Problems

1. Solve Problem 1 of Art. 21 graphically to the scale of 1 inch = 5 feet.
2. Solve Problem 2 of Art. 21 graphically to the scale of 1 inch = 5 feet.
3. Solve Problem 4 of Art. 21 graphically to the scale of 1 inch = 5 feet.

24. Product of Two Vectors.—There are two ways of multiplying two vectors. The result of one method is a *vector* and is called the *vector product* of the two vectors. The result of the other method is a mere number without direction, and is called *scalar product* of the two vectors.

If the length of vector **a** is a units and the length of vector **b** is b units, and if the angle between the two vectors is θ,

$$\text{scalar product } \mathbf{a}.\,\mathbf{b} = ab \cos \theta. \qquad \text{Formula I.}$$

Since $b \cos \theta$ is the length of the orthographic projection of vector **b** upon the line of vector **a**, the scalar product of two vectors may be defined as numerically equal to the product of the magnitude of one vector multiplied by the magnitude of the projection of the other vector upon its direction. Since

$$ab \cos \theta = ba \cos \theta,$$

either vector may be regarded as projected upon the other. When the angle between the two vectors is zero, their scalar product is simply the product of their magnitudes. Scalar multiplication of vectors, when they are in the same direction, is equivalent to multiplication of ordinary arithmetic, just as addition of vectors in the same direction is equivalent to addition of ordinary arithmetic.

The work done by a force, which is the product of the magnitude of the force multiplied by the component of the displacement in the direction of the force, is an example of a scalar product.

The *vector product* of two vectors **a** and **b** is defined by the equation

$$\text{vector product } \mathbf{a} \times \mathbf{b} = ab \sin \theta. \qquad \text{Formula II,}$$

The vector product of two vectors is numerically equal to the product of the length of one vector multiplied by the component of the other vector perpendicular to its direction.

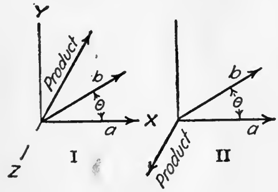


FIG 26.

The vector product of two vectors is defined as a vector perpendicular to the plane of the two vectors which are multiplied together. In the right-handed system of coördinates, the direction of the angle between the vectors and the direction of the vector product bear the same relation to each other as the direction of rotation and motion of a right-handed screw. If the rotation from **b** to **a** in the XY plane of Fig. 26, I, is clockwise, the vector product $\mathbf{b} \times \mathbf{a}$ is along the Z axis away from the observer. If the rotation from **a** to **b** is counter-clockwise, the vector product $\mathbf{a} \times \mathbf{b}$ is directed along the Z axis toward the observer.

The moment of a force, which is the product of the force multiplied by the length of the component of the moment arm perpendicular to the direction of the force, is an example of a vector product.

There are several ways of indicating the multiplication of two vectors. The most convenient one is that used by Gibbs:

$$\text{Scalar product of } \mathbf{ab} = \mathbf{a}.\mathbf{b},$$

which is read *a dot b* and is called the *dot product.*

$$\text{Vector product of } \mathbf{ab} = \mathbf{a} \times \mathbf{b},$$

which is read *a cross b* and is called the *cross product.*

This method of writing the vector products has not come into general use and is not employed in elementary texts on mechanics.

Problems

1. Given vector $\mathbf{a} = 6$ feet at 20°, vector $\mathbf{b} = 4$ feet at 45°, find $\mathbf{a}.\mathbf{b}$, $\mathbf{a} \times \mathbf{b}$, and $\mathbf{b} \times \mathbf{a}$.

Ans. $\mathbf{a}.\mathbf{b} = 21.75$; $\mathbf{a} \times \mathbf{b} = 10.14$ ft. toward the front; $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$.

2. A vector $\mathbf{a} = 16.4$ feet at 0°, a vector $\mathbf{b} = 20.4$ feet at 25°, and a vector $\mathbf{c} = 18.2$ feet at 65°. All these vectors are in one plane. Find $\mathbf{a}.\mathbf{b}$, $\mathbf{a}.\mathbf{c}$, $\mathbf{a} \times \mathbf{b}$, $\mathbf{a} \times \mathbf{c}$, $\mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{b}$, $\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.

Ans. $\mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{b} = -97.27$ ft.; $\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = 411.9$ ft.

25. Summary.—The magnitude of a quantity may be represented by a number or by the length of a line.

A quantity which has *magnitude* and *direction* is called a *vector.* A vector is represented by a line.

A vector in a plane may be expressed by two numbers. One number gives its length and the other gives its angle with a known line. A vector in a plane may also be expressed by means of its projections on two lines.

A vector in space may be expressed by three numbers. Its length may be given and its angle with two lines at right angles with each other. These angles are called direction angles and their cosines are direction cosines of the vector. The direction angle with the third axis at right angles to the plane of these two lines may be found by the relation of analytic geometry that the sum of the squares of the three direction cosines is equal to unity.

The direction of a vector in space may also be found by means of its angle with one axis and the angle which the plane through the vector and this axis makes with another reference plane through the axis. This method is equivalent to co-latitude and longitude and is sometimes called representation by spherical coördinates. A vector in space may also be expressed by means of its projections on three axis, only two of which are in any one plane.

The sum of two vectors is obtained by placing the origin of the second vector at the terminus of the first vector. The required sum is the vector which joins the origin of the first vector with the terminus of the second. The vector sum is called the *resultant* vector.

Vectors which added together form a given vector are called *components* of the given vector. The most important components are those which are obtained by orthographic projection.

To *calculate* the vector sum of several vectors in space, each vector is first resolved into its components along the three coördinate axes. The sum of the components along any axis forms one edge of a rectangular parallelopiped. The resultant vector is the diagonal of this parallelopiped. If the vectors are in a single plane, the parallelopiped is reduced to a rectangle, and the resultant vector is the hypotenuse of the right-angled triangle which forms one half of this rectangle.

To subtract a vector add an equal vector in the opposite direction.

There are two kinds of products of vectors. The *vector* product is the product of the length of the vectors multiplied by the sine of the angle between them. This product is a vector and is normal to the plane of the given vectors. The *scalar* product is the product of the length of the vectors multiplied by the cosine of the angle between them. This p oduct is a scalar quantity. It has magnitude but not direction.

CHAPTER III

APPLICATION OF FORCE

26. Tension, Compression, and Shear.—Figure 27 shows a 10-pound mass (a so-called 10-pound weight) supported by a cord, which is fastened to a short horizontal beam at the top. The beam pulls upward on the cord at the top, and the 10-pound mass pulls downward at the bottom. The cord is said to be in *tension*. A body is in tension when it is subjected to a pair of equal forces which are along the same line, opposite in direction, and *away* from each other. A body in tension is said to be under *tensile stress.*

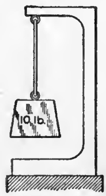


Fig. 27.

Figure 28 shows a 50-pound mass resting on a short block. The block is subjected to a downward push of 50 pounds at the top, and an upward push of 50 pounds (in addition to its own weight) from the support at the bottom. The block is in *compression* and is subjected to *compressive stress.* A body is in compression when it is subjected to a pair of equal forces which are along the same line, opposite in direction, and *toward* each other.

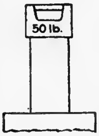


Fig. 28.

In Fig. 29, the weight tends to slide the right portion of the block downward relatively to the left portion. The block *AB* is said to be in *shear* and subjected to *shearing stress.* A body is in shear when it is subjected to a pair of equal forces which are opposite in direction, and which *act along parallel planes.*

A body in tension is lengthened along the direction of the forces; a body in compression is shortened.

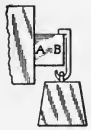


Fig. 29.

27. States of Matter.—Materials exist in one of three forms, solid, liquid, or gaseous.

A block of steel or a block of ice is an example of matter in the *solid state.* A solid may be supported against the force of gravity by a single force in one direction. A block of ice resting on a table is supported by the reaction of the table, which is a single force acting upward. A flexible cord, on the other hand, can not be made to stand on the lower end, but may be supported on the side, or suspended from the upper end, and, in these ways, fulfill the condition of support by a single force in one direction.

If a block of ice is heated and changed into the *liquid state,* it can no longer be supported by a single force upward, but must be confined on all sides by walls which exert lateral pressure.

If the water is heated still further, so as to turn it into steam, it comes into the *gaseous state.* It must now be confined at the top, as well as at the sides and bottom, to prevent it from expanding.

A solid has a definite form and a definite volume; a liquid has a definite volume but not a definite form; a gas has neither a definite form nor a definite volume.

A solid can resist tension, compression, or shear. A liquid offers no resistance to shear and will support practically no tension. If confined so as to prevent lateral shear, a liquid or gas will resist compression. The volume of a gas is greatly changed by a small change of the compression; the volume of a liquid is changed very little. A *viscous* liquid offers some resistance to shear, especially when the force is applied for a short time. Tar at ordinary temperatures, and steel at red heat are viscous. Every liquid has some viscosity. It is, however, very small in the case of water or alcohol, and relatively large in heavy oils. A liquid with absolutely no viscosity is called an ideal perfect fluid.

28. A Rigid Body.—A solid body, which suffers little change of form when subjected to considerable force, is called a rigid body. All bodies are elastic, so that there is some change in form or dimensions when force is applied; but these changes are frequently so small as to be negligible, except for measurements requiring the greatest accuracy. Such bodies are regarded as rigid. Fig. 30, I, shows a beam supported at the middle with a load on each end. The beam is somewhat bent, but the amount of bending may be so small that the distance of the loads from the vertical line through the support is not materially changed. Fig. 30, II, shows a lighter beam, in which the amount of bending is sufficiently great to change materially its form and dimensions. If, however, the beam be considered as it is now loaded, it may be

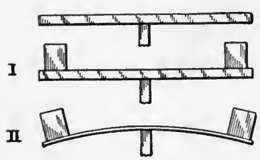


FIG. 30.

regarded as a rigid body, but one of different form and dimensions from that which it would have with a different loading.

In elementary calculations of Mechanics no allowance is made for slight elastic deformations. In that branch of Mechanics which is called Strength of Materials or Mechanics of Materials, however, these deformations are taken into account.

29. A Flexible Cord.—Fig. 31,I, shows a *stiff* rope supported at the middle. This rope has some rigidity. It is similar to a very flexible beam. Fig. 31, II, shows a *perfectly flexible* cord. The

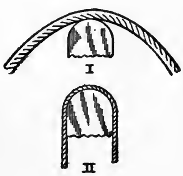


FIG. 31.

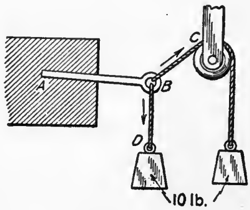


FIG. 32.

cord takes the form of the support at the top and hangs vertically downward at the ends. An ideal flexible cord offers no resistance to bending. A flexible cord or rope can exert force only in the form of tension in the direction of its length. When a flexible cord forms a part of a structure or piece of apparatus, the position and direction of the force which it exerts is definitely known.

In Fig. 32, two flexible cords are attached to the beam AB. The direction of the tension exerted by each cord on the beam is shown by the arrow; the position of the force in each is known to be along its axis. On the other hand, the direction of the force exerted by the beam at B can not be determined by inspection of the diagram, but must be calculated from the direction and magnitude of the forces in the cords.

30. Equilibrium.—Fig. 33 shows a 10-pound mass (an ordinary 10-pound weight) supported by a flexible cord. If the mass is stationary with respect to the earth, that is, if it is not swinging as a pendulum or vibrating up and down on an elastic support, it is said to be in *equilibrium*. In order that a body may be in equilibrium, the forces which act on it from other bodies must *balance*. Instead of stating that the body is in equilibrium, it is frequently said that the *forces* which act on the body are in equilibrium. A

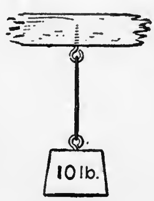


FIG. 33.

body in equilibrium is not necessarily stationary. It may be moving with constant speed in a straight line.

Two forces act on the 10-pound mass of Fig. 33. These are the pull of the cord directed upward, and the pull of gravity, acting from the earth through the ether, directed downward. These two forces are equal in magnitude; that is, each is a force of 10 pounds. They are opposite in direction and are exerted along the same vertical line.

A body is in equilibrium when the forces in any direction are balanced by the forces along the same line in the opposite direction.

The attraction of the earth and the pull on the cord in Fig. 33 *do not* constitute action and reaction, as is often erroneously assumed. There are two sets of action and reaction involved in this equilibrium. The pull of the earth on the body and the pull of the body on the earth form one action and reaction. The pull of the cord on the body and the pull of the body on the cord form the other.

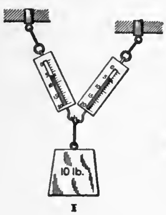


FIG. 34.

In Fig. 34, the knot at which the three cords meet may be regarded as the body in equilibrium. The downward pull of the cord from the 10-pound mass is balanced by the combined upward pulls of the cords from the spring balances.

31. A Smooth Surface.—A smooth surface is one which offers no resistance to the motion of a body along it. The only force which a smooth body can exert is normal to its surface. A body in motion on a smooth, *horizontal* surface will continue to move for an indefinite time with no diminution of speed.

No surface is perfectly smooth; there is always some *friction.* Friction is a force at the surface of contact of two bodies which resists the motion of one body along the surface of the other. The friction between smooth ice and a polished steel runner, or the friction between a metal shaft and a well lubricated bearing, is small.

Figure 35 represents a horizontal surface upon which a body B is placed. A flexible cord, which runs over a smooth pulley and supports a weight P is attached to B. If there were no friction at the pulley, the tension in the cord at B would be equal to the weight of P. With some friction, the tension P' is slightly less than the weight of P. If the force P' parallel to the surface is just sufficient to start the body in motion, this force is said to be equal to the *starting friction* between the body and the surface upon which it rests. If the force P' is just sufficient to keep the body moving with uniform speed after it has been started by some additional force, then the force P' is said to be equal to the *moving friction*. The force of friction is represented in Fig. 35 by the single-barbed arrow. This arrow indicates that the friction from the lower surface to the body B is directed toward the left. This is opposite to the direction of the force P'. The friction of the body B upon the surface below it is directed toward the right.

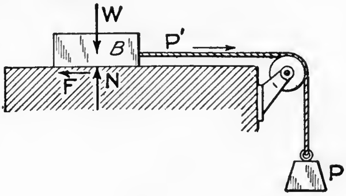


FIG. 35.

For the purpose of demonstrating the mechanics of ideal frictionless surfaces, the moving body may be provided with wheels or rollers, as in Fig. 36. This figure approximates closely to the ideal condition of a smooth bar which rests on a smooth horizontal floor and leans against a smooth vertical wall.

The subject of friction is continued in Chapter XII. For the present, bodies will be assumed to be frictionless. The condition of equilibrium for perfectly smooth bodies is approximately midway between the limiting conditions of equilibrium for rough bodies. A 10-pound mass on a *smooth plane*, which makes an angle of 30 degrees with the horizontal, may be held in equilibrium by a force of 5 pounds parallel to the plane. If the plane were not smooth, the body would still be held in equilibrium by the 5-pound force, or by a force somewhat greater or somewhat less than 5 pounds. If the coefficient of friction were 0.1, as in Problem 10 of Art. 112, the body would be held by any force parallel to the plane between the limits of 4.134 pounds and 5.866 pounds. The same mass may be held on a *smooth plane* by a *horizontal* force of 5.77 pounds; and may be held on a plane for which the coefficient of friction is 0.1 by any horizontal force between the limits of 4.74 pounds and 7.19 pounds (Problem 11 Art. 112). In the first case, the force which holds the body on the smooth plane is exactly midway between the limiting forces for the rough plane; in the second case, it deviates a little from the median value.

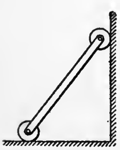


FIG. 36.

32. A Smooth Hinge.—Two bodies are frequently connected as in Fig. 37. A cylindrical pin passes through cylindrical holes in each of the bodies B and C. The pin may be fixed in one of the bodies but must be free to turn in the other. This form of connection is called a *hinge*. One body can rotate relatively to the other in a plane which is perpendicular to the axis of the pin. In a *smooth* hinge or pin-connection the force between the pin and the hollow cylinder is normal to the curved surfaces at the line of contact. The force is, therefore, along a line through the axes of the pin and the hollow cylinder, and in a plane normal to these axes.

In Fig. 37, the hollow cylinder in body C is drawn much larger than the pin A. In practice the pin is made to fit the cylinder with little clearance.

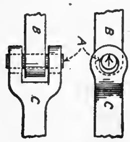


FIG. 37.

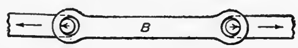


FIG. 38.

A body with a smooth hinge at each end, as in Fig. 38, is called a *link*. A link transmits force in the direction of the line joining the two hinges.

33. Resultant and Equilibrant.—In Fig. 34, the upward pull of the vertical cord is replaced at the knot A by the combined pull of the cords which lead to the spring balances. The upward pull in the vertical cord is equal to the *resultant* of the forces in the other two cords. The downward pull of the vertical cord at the knot, which balances the combined upward pull of the other two cords, is called the *equilibrant*. The *resultant* of a combination of forces is the single force whose effect on the equilibrium of a body is equivalent to that of the combination of forces. The *equilibrant* of a combination of forces is the single force which will balance these forces and produce equilibrium. The resultant of a set of forces is equal and opposite to their equilibrant. Though the word *equilibrant* is not in general use, the distinction between equilibrant and resultant must be kept clearly in mind to avoid confusion.

In Art. 29, it was stated that the force in each flexible cord of Fig. 32 is transmitted along the axis of the cord. In reality, the force is transmitted in all parts of the cord. The *resultant* of all these forces is transmitted along the geometrical axis. In Art. 30, it was stated that the pull of gravity on the 10-pound mass is vertically downward. In reality, every particle of the 10-pound mass is attracted by every particle of the earth, so that some of the force is nearly horizontal. The *resultant* of all these forces is vertically downward.

The resultant pull of gravity on a body passes through a point which is called the *center* of *gravity* or *center* of *mass* of the body.

Any system of forces in equilibrium may be divided into two groups. The resultant of the forces of one group forms the equilibrant for the resultant of the forces of the other group. In Fig. 34, the pull from the left spring balance may be regarded as combined with the downward force of 10 pounds. The resultant of these two forces is equal and opposite to the pull from the right spring balance. In like manner, the pull from the right spring balance may be combined with the downward force of 10 pounds. The resultant of these two forces is balanced by the pull from the left spring balance. The force in any one of the cords of Fig. 34 serves as the equilibrant which balances the resultant of the forces in the other two cords.

The forces which make up a resultant force are called *components* of the resultant.

In the calculation of problems of equilibrium, it is often possible to use the resultant of a number of forces, instead of the separate forces, and, in this way, to make the computations more simple, without introducing any error in the result.

34. The Force Circuit.—In Fig. 27, the earth pulls downward on the 10-pound mass. The force is transmitted from the earth through the ether. The 10-pound mass pulls down on the horizontal beam. The force is transmitted from the mass to the beam in the form of tension in the cord. The beam pushes down on the top of the post *B C*. The force is transmitted from the top of the cord to the top of the post in the form of shear in the horizontal beam. The force is transmitted as compression from the top to the bottom of the post. It is next transmitted, toward the left, in the base as shear. Finally it passes to the earth as compression, and the resultant compression is vertical and directly underneath the center of gravity of the 10-pound mass.

35. The Free Body.—In a problem of equilibrium the forces are considered which act on some definite body or portion of a body. This body or portion of a body is called the *free* body in equilibrium. In Fig. 27, the 10-pound mass may be taken as the free body. It is in equilibrium under the pull of the earth downward and the pull of the cord upward. The cord may likewise be taken as the *free body* in equilibrium under the pull of the 10-pound mass and the pull of gravity on its own mass, both of which are balanced by the upward pull of the beam *A B*. The entire system of Fig. 27 may be taken as a free body. In this case the weights of the parts of the system form one set of external forces. These are balanced by the upward reaction at the support.

In Fig. 35, the mass *B* may be regarded as the *free body.* The forces which act on it are, (1) the force of gravity downward, (2) the vertical reaction of the plane upward, (3) the horizontal friction of the supporting plane toward the left, and (4) the horizontal pull of the rope toward the right. The force of gravity, or weight, is represented by the arrow marked **W.** The upward reaction of the plane is represented by the arrow marked **N.** Generally the arrow which represents the weight is drawn downward *from* the body; in this case it is placed above the body and drawn downward *to* it. This is done in order to avoid confusion with the upward reaction **N,** which would fall on the same line if both were drawn below the body. The friction from the supporting plane to the body is represented by the single-barbed arrow **F;** and the pull from the cord by the arrow **P′.** The arrows must show the direction of the force exerted by the other bodies or parts of a body upon the *free body.* If *B* is the free body, **F** is toward the left and **N** is upward. If the supporting plane were under consideration as the *free body,* then the friction **F** from *B* to the plane would be toward the right, and the normal **N** from *B* to the plane would be downward.

In considering a problem of equilibrium, it is absolutely necessary to decide what body or portion of a body is to be taken as the *free body,* and then to designate all the forces which act on that *free body.*

36. Application of Forces.—In Fig. 34, the three cords meet at a point at the knot *A*. Forces which meet at a point are *concurrent.* All these cords lie in one vertical plane. Forces in the same plane are *coplanar.* Fig. 34 is an illustration of *concurrent, coplanar* forces.

In Fig. 39, the three forces do not meet at a point. They are *non-concurrent.* All these forces lie in one plane. Fig. 39 is an illustration of *non-concurrent, coplanar* forces.

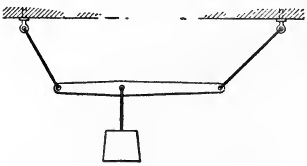


FIG. 39.

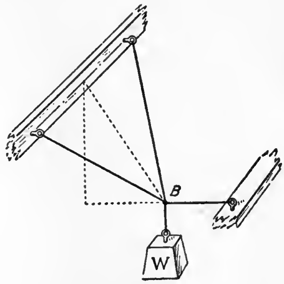


FIG. 40.

In Fig. 40, there are four forces which meet at the point *B*. They are *concurrent.* They do not all lie in one plane. They are *non-coplanar.* Fig. 40 is an illustration of *concurrent, non-coplanar* forces.

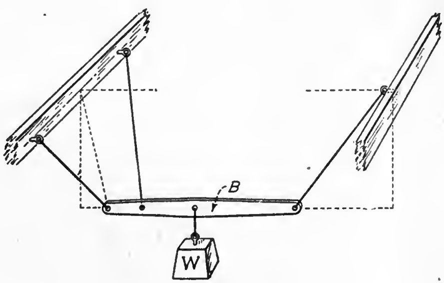


FIG. 41.

In Fig. 41, the forces which act on the horizontal bar *B* do not meet at one point and do not lie in one plane. Fig. 41 is an illustration of *non-concurrent, non-coplanar* forces.

These four classes of the application of forces will be considered separately in the chapters which follow.

37. Resultant of Concurrent Forces.—A force has magnitude and direction and a definite position. A vector has magnitude and direction, so that it is natural to assume that a force may be represented by a vector and that the resultant of two concurrent forces may be represented by their vector sum. Experiments show that this assumption is true, so that it may be stated as an axiom, which has been amply verified by measurements: *If* *two concurrent forces are represented by vectors, the resultant of the forces is represented, in direction and magnitude, by the vector sum of these two vectors.* The two vectors and their resultant form a triangle, Fig. 42, II, which is called the *force triangle.* The resultant force passes through the point at which the two forces are concurrent,—the point *O* of Fig. 42, I.

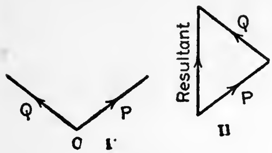


FIG. 42.

Since the resultant of two concurrent forces may be represented by their vector sum, the resultant of three concurrent forces may be represented by the vector sum obtained by combining the resultant of the first two forces with the third force. The third force may not be in the plane of the first two forces. When the three forces *are* all in one plane, it is not necessary to draw the resultant of the first two forces and then to draw the third force. All three forces may be added at one time, as in Fig. 43. The figure thus obtained is called the *force polygon.*

The vector sum of two vectors **a** and **b** may be drawn by adding vector **b** to vector **a**, or by adding vector **a** to vector **b**, as shown in Fig. 11. If the addition is made in both orders, each starting from the same origin, the figure obtained is a parallelogram. For this reason, the method of finding the resultant of two forces by means of their vector sum is often called the *parallelogram law.* It is, however, entirely unnecessary to draw the parallelogram. The single triangle, which is one-half of the parallelogram, is sufficient. Moreover, the use of parallelograms causes confusion when three or more forces are involved in the problem. It is best, therefore, to consider the force triangle and the force polygon, and to forget the parallelogram.

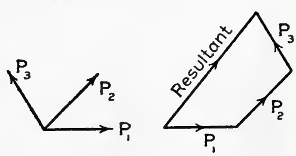


FIG. 43.

The process of finding the resultant of two or more forces is called the *composition* of forces, and the forces which make up the resultant are called its *components.*

38. Resolution of Forces.—The process of finding the components of a force is *resolution* of the force. The force is said to be *resolved* into its *components.* The most important case of resolution is that in which the components are at right angles to each other. There may be two components at right angles to each other in the same plane, or three components, any one of which is perpendicular to the plane of the other two. When the word component is used, unless otherwise stated, a component of this kind is meant. Frequently only one component is considered. Fig. 44 shows a force **P** at an angle with the line *AB*. The component of the force **P** in the direction of the line *AB* is equal in length to the orthographic projection upon *AB* of the vector which represents the force.

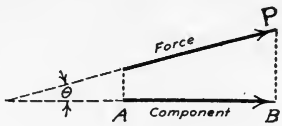


FIG. 44.

The process of finding the orthographic component of a force in a given direction is called resolution in that direction.

39. Work.—Figure 45 represents a body *B* acted on by a force **P**. The force is applied at the point *C*, which is called its point of application. The force may be exerted through a flexible cord, which is tied to the body at the point of application. The body moves to a second position while the force continues to act in the same direction. The point *C* moves a distance of s units of length. This is the *displacement* of the point of application. If the body does not rotate, the displacement of all parts is the same. If, however, there is any rotation, the displacement of the point of application may be different from that of other parts of the body.

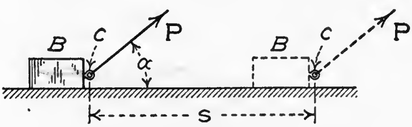


FIG. 45.

If the displacement of the point of application of the force makes an angle α with the direction of the force, the work done by the force is defined by the equation,

$$\text{work} = P \cos \alpha\ s \qquad \text{Formula III.}$$

Since $P \cos \alpha$ is the component of the force in the direction of the displacement, the definition of work may be stated as follows: *The work of a force is the product of the displacement of its point of application multiplied by the component of the force in the direction of the displacement.*

$$P \cos \alpha\ s = P\ s \cos \alpha.$$

From the second member of this equation, work may be defined as the product of the force multiplied by the component of the displacement in the direction of the force.

If the force is expressed in pounds and the displacement in feet, the work is in foot-pounds. Other units of work are inch-pounds, gram-centimeters, kilogram-meters, and ergs.

Both force and displacement are vectors. The product of two vectors may be a third vector, or may be a *scalar* quantity, which has magnitude but not direction. Work is a scalar product, and is not a vector.

40. Classes of Equilibrium.—There are three kinds of equilibrium. These are called *stable*, *unstable*, and *neutral* or *indifferent*.

Figure 46 shows three cases of *stable* equilibrium. When a body in stable equilibrium is displaced slightly from its position of

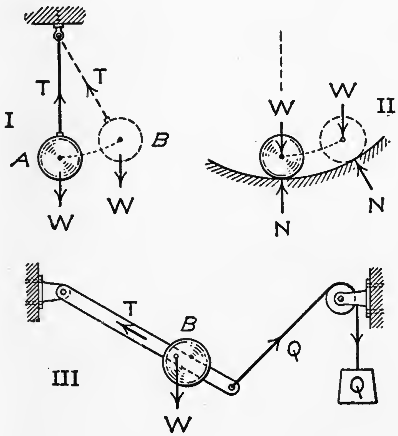


FIG. 46.

equilibrium by an additional force, it will return to that position of equilibrium when the additional force ceases to act. In Fig. 46, I, the body is hung on a flexible cord. The body is in stable equilibrium in position *A*. The cord is vertical and the center of mass is directly under the point of support. If an additional force is applied, which deflects the body to position *B*, it will return to position *A* when the additional force is removed. In the position of equilibrium the forces which act on the body are its weight and the tension in the cord. When the body is moved from position *A* to position *B*, no work is done by the tension in the cord, since the displacement is perpendicular to the direction of the force; negative work is done by the weight, since the upward component of the displacement is opposite the direction of the force of gravity. When a body in stable equilibrium is displaced slightly, the original forces which produce equilibrium do *negative work*, and positive work must be done by the additional forces to cause the displacement.

Figure 46, II, shows a body on a smooth, curved surface which is concave upward. The forces which produce equilibrium are the weight of the body and the normal reaction of the surface. If the surface is that of a sphere or circular cylinder, the conditions are the same as those of a body suspended by a flexible cord.

Figure 46, III, shows a body suspended from a smooth hinge and attached to a cord which runs over a smooth pulley and carries a second body Q. If the body is displaced toward the right, the weight W does negative work and the pull in the cord does positive work. It may be shown that the negative work is the greater, and that the equilibrium is stable.

Figure 47 shows three cases of unstable equilibrium. Figure 47, I, shows a body on a smooth curved surface which is concave

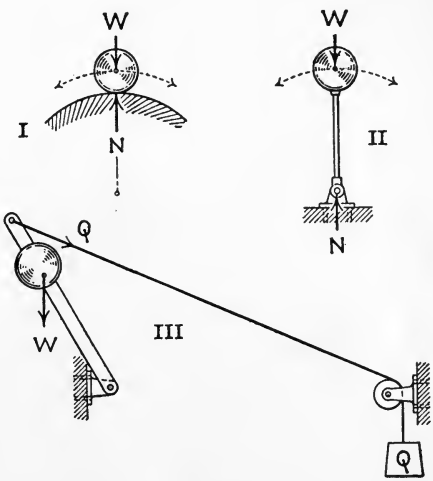


FIG. 47.

downward. If the body is moved slightly in either direction from the position of equilibrium, it will not return, but will continue to move in that direction. The vertical component of the displacement is in the *same* direction as the force of gravity; hence its weight does positive work when it is moved from the position of equilibrium.

Figure 47, II, shows a body supported on a smooth hinge which directly under its center of mass. The motion is the same as that of Fig. 47, I, when the surface is cylindrical. The body of Fig. 47, II, will rotate about the hinge until its center of mass is directly under the point of support. It will then be in stable equilibrium in the position of Fig. 46, I. In like manner, the body of Fig. 47, I, will move to the position of Fig. 46, II, provided the surface is continuous, and there is some arrangement to keep it from falling off when below the center.

Figure 47, III, is similar to Fig. 46, III, but the equilibrium is unstable. If displaced, it will rotate to the position of stable equilibrium of Fig. 46, III.

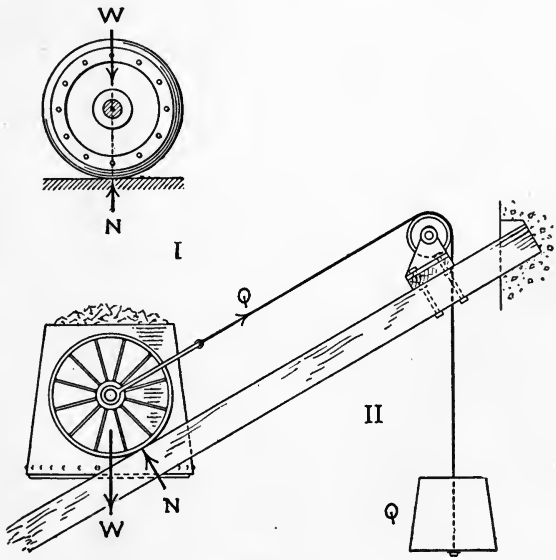


FIG. 48.

Figure 48 shows two cases of *neutral equilibrium*. The body remains in equilibrium in any position. Figure 48, I, represents a body on a smooth, horizontal plane surface. No work is done when it is displaced, as the weight and the normal reaction of the surface are both perpendicular to the direction of the displacement.

Figure 48, II, shows a body on a smooth inclined plane. The body is supported by a cord which runs parallel to the plane and exerts a constant pull Q. If the body is displaced on the plane, the positive work of the force Q is equal to the negative work of the weight. If the body is displaced down the plane, the positive work of the weight is equal to the negative work of the force Q.

CHAPTER IV

CONCURRENT CO-PLANAR FORCES

41. Resultant.—If the direction and magnitude of each of several forces is represented by a vector, the direction and magnitude of the resultant of these forces will be represented by the vector sum of these vectors. As stated in Art. 37, this statement may be regarded as an axiom which has been amply verified by experiment. If these forces are concurrent at a given point, their resultant will pass through this point.

In Fig. 49, I, two flexible cords in a vertical plane are attached to a beam at a point A. Each cord runs over a smooth pulley.

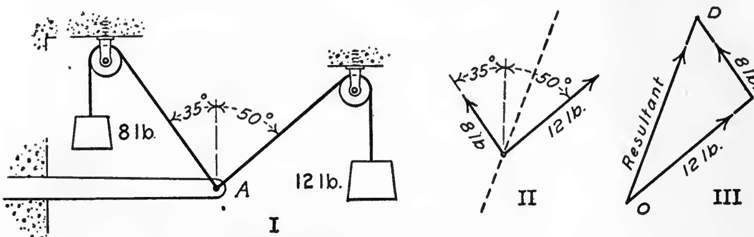


FIG. 49.

The left cord makes an angle of 35 degrees to the left of the vertical, and supports a mass of 8 pounds; the right cord makes an angle of 50 degrees to the right of the vertical, and supports a mass of 12 pounds. If the pulleys are frictionless, the tension of each cord at A will be equal to the mass which it supports. Fig. 49, II, is the *space diagram* representing the direction and location of these forces, which are concurrent at A. Figure 49, III, is the *force polygon*, which, in this case, is a *triangle*. To construct the force triangle, point O is selected as the origin and a line is drawn through this point parallel to the direction of the force of 12 pounds in the space diagram. A length of 12 units, from O to C, is measured in this line. The point C is the terminus of the vector which represents the 12-pound force. From C as the origin, a second vector, 8 units in length, is drawn parallel to the direction of the force of 8 pounds in the space diagram.

The *direction* and *magnitude* of the resultant of these two forces is given by the line $O D$ which is drawn from the origin of the first vector to the terminus of the second vector. Finally, through the point of application of the forces on the space diagram, a broken line is drawn parallel to the direction of the resultant OD of the force triangle. This broken line gives the *location* of the line of action of the resultant force.

Problems

1. Find the magnitude and direction of the resultant of the following forces in a horizontal plane: 20 pounds north 75 degrees east, 15 pounds north 20 degrees east, and 8 pounds north 25 degrees west. Solve graphically to the scale of 1 inch = 5 pounds.

2. Two ropes in a vertical plane are attached to a fixed point. One rope, which supports a mass of 40 pounds on the free end, runs over a smooth pulley located 16 feet higher than the fixed point, and 7 feet to the left of the vertical line through it. The second rope, which supports a mass of 35 pounds, runs over a smooth pulley located 8 feet higher than the fixed point, and 15 feet to the right of the vertical line through it. Construct the space diagram to the scale of 1 inch = 5 feet. Then construct the force triangle to the scale of 1 inch = 10 pounds. Measure the resultant in the force triangle and express its magnitude in pounds and its angle with the vertical in degrees and minutes. Draw a broken line in the space diagram to show the line of action of the resultant force.

42. Calculation of Resultants and Components.

—The resultant of *two* forces may be calculated by means of the solution of the force triangle.

Problems

1. Find the resultant of a force of 24 pounds and a force of 30 pounds which makes an angle of 35 degrees with the direction of the 24-pound force. Construct the force triangle and solve for the *magnitude* of the resultant by means of the law of cosines. After the magnitude is found, calculate the angle by means of the law of sines. Check all results by means of the projections of the forces upon the line of action of the resultant.

2. A force of 234 pounds is north 24 degrees west, and a concurrent force of 256 pounds is north 17 degrees east. Construct the force triangle and solve for the *direction* of the resultant by means of the ratios of the sum and difference of the sides and the tangents of one-half the sum and one-half the difference of the angles opposite these sides. After all the angles are found, calculate the side which represents the resultant by means of the law of sines.

3. A force **P** makes an angle of 43 degrees to the right of the vertical. A force **Q**, concurrent with it, has a magnitude of 84.26 pounds. The resultant of these forces is a vertical force of 116.45 pounds. Find the direction of the force **Q** and the magnitude of the force **P**, using logarithms. Construct the force triangle to the scale of 1 inch = 20 pounds and compare with the calculated results.

4. The resultant of two forces of 16 pounds and 24 pounds in a horizontal plane is a force of 34 pounds north 10 degrees east. Find the direction of the forces by means of the formula for the tangent of the half angle.

When it is desired to find the resultant of more than two forces, the method of calculation by means of triangles is laborious. To solve such problems each force is resolved along the direction of two axes which are at right angles to each other. The sum of the components of all these forces along one axis is the component of the resultant along that axis, and the sum of the components of all these forces along the other axis is the component of the resultant along that axis. These two components of the resultant form two sides of a right-angled triangle of which the resultant is the hypotenuse. When the resultant of several forces is to be found, the work may be arranged in tabular form, as was shown in Problem 4 of Art. 17.

Problems

5. A horizontal force of 25 pounds is north 37 degrees east. Find the east and north components.

Ans. East component = 15.05 lb.; north component = 19.97 lb.

6. A horizontal force of 46 pounds is north 22 degrees west. Find the component north and the component east.

Ans. East component = −17.23 lb.; north component = 42.65 lb.

7. A force of 24 pounds makes an angle of 20 degrees with the horizontal. A concurrent force of 30 pounds in the same vertical plane makes an angle of 64 degrees with the horizontal on the same side of the vertical. Find the sum of the horizontal and the sum of the vertical components.

Ans. H = 35.70 lb.; V = 35.17 lb.

8. Find the direction and magnitude of the resultant of the two forces of Problem 7.

9. Solve Problem 8 by resolving along the line of the force of 24 pounds and along a line perpendicular to the force of 24 pounds in the same vertical plane.

10. Find the direction and magnitude of the resultant of 14 pounds north 27 degrees east, 15 pounds north 35 degrees west, and 18 pounds south 56 degrees west. Resolve east and north then check by resolutions along some other pair of axes.

11. Solve Problem 1 of Art. 41 by means of resolutions, and compare the results with the graphical solution.

12. Solve Problem 2 of Art. 41 and compare the results with the graphical solution.

13. Find the direction and magnitude of the resultant of 24.2 pounds at 20 degrees, 17.8 pounds at 65 degrees, 22.5 pounds at 110 degrees, 12.6 pounds at 165 degrees, and 31.4 pounds at 234 degrees. Resolve along 0 degrees and 90 degrees and check by resolutions along 20 degrees and 110 degrees.

43. Equilibrium.—When a body is in equilibrium under the action of two forces, these forces are equal, opposite, and along the same line. If a body is in equilibrium under the action of three forces, the resultant of any two of these forces must be equal and opposite the third force, and must act along the same line. Figure 50 shows three flexible cords attached to a point. The resultant of the forces **P** and **Q** is the force **R**. The direction of **R** is opposite the direction of the third force **T**. Figure 50, II, shows the force triangle for the resultant.

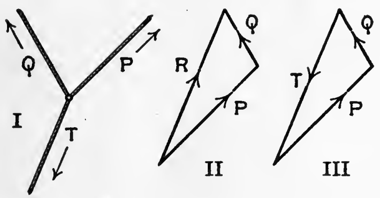


FIG. 50.

Figure 50, III, shows the triangle for the three forces in equilibrium. The only difference between Fig. 50, II, and Fig. 50, III, is the direction of one line. The *resultant* in Fig. 50, II, extends from the origin of the vector **P** to the terminus of the vector **Q.** The *equilibrant* in Fig. 50, III, extends from the terminus of sector **Q** to the origin of vector **P.** This last vector **T** is called the *closing line.*

When a body is in equilibrium under the action of several concurrent forces, the resultant of all the forces is zero, as may be seen from Fig. 50, III. The arrows which represent the direction of the forces follow each other around the force diagram from the origin of the first vector to the terminus of the last vector. The arrow on one side of each angle of the diagram points toward the angle and the arrow on the other side of the angle points away from it. The terminus of the last vector coincides with the origin of the first. It is customary to say that the force polygon closes when equilibrium exists.

To solve a problem of equilibrium of concurrent forces, that part of the force polygon which represents the *known* forces is constructed first. If there is only one *unknown* force, it is represented in the force polygon by the closing line, which is drawn from the terminus of the last known force to the origin of the first one. If there are two forces of *known direction* but of *unknown magnitude*, a line is drawn through the terminus of the last known vector in the direction of one of these unknown forces, and a line is drawn through the origin of the first known vector in the direction of the other unknown force. The length of each of these lines to their point or intersection represents the required force.

Example

A 40-pound mass rests on a smooth inclined plane, which makes an angle of 35 degrees with the horizontal. It is supported by a horizontal push of 12 pounds, and a pull *P* parallel to the plane. Find the force *P* and the normal reaction.

Figure 51, I, is the space diagram showing the direction of all the forces. The mass of 40 pounds is the *free body* in equilibrium. The arrows show the direction of the forces which act from other bodies on the *free body*. The reaction of the plane is upward at an angle of 35 degrees with the vertical. The pull of the earth on the *free body* is a downward force of 40 pounds. The length of the lines in this diagram is not necessarily proportional to the magnitude of the forces.

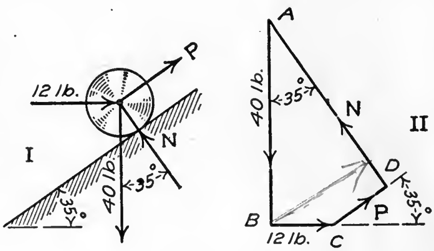


FIG. 51.

Figure 51, II, is the force polygon. The known forces are 40 pounds downward and 12 pounds horizontal toward the right. As the first step in the construction of the force polygon lay off *AB* 40 units in length. From *B* draw the horizontal line *BC* 12 units in length. The arrow in *AB* is downward *toward B*, and the arrow in *BC* is to the right, directed *away from B*. Through *C* draw a line parallel to the force **P** of the space diagram, and through *A* draw a line parallel to the direction of the normal reaction. Extend these two lines till they intersect at *D*. The length of *CD* measures the tension parallel to the inclined plane, and the length of *DA* represents the normal reaction of the plane against the 40-pound mass. The arrow in *BC* is toward *C*. The arrow in *CD* must by away from *C* toward *D;* the corresponding direction in the space diagram is up the plane. The arrow in *DA* is from *D* toward *A*.

Problems

1. Solve the example above graphically to the scale of 1 inch = 10 pounds, with the horizontal force 20 pounds instead of 12 pounds.

2. A 20-pound mass on a 35 degree inclined plane is held in equilibrium by a horizontal push of 5 pounds, and by a pull at an angle of 40 degrees with the horizontal and an angle of 5 degrees with the plane. Solve for this pull and the normal reaction to the scale of 1 inch = 5 pounds.

3. A 50-pound mass is supported by 3 cords in the same vertical plane. One cord makes an angle of 45 degrees to the left of the vertical. The second cord makes an angle of 30 degrees to the right of the vertical. The third cord runs horizontally toward the right and exerts a pull of 10 pounds. Find the tension in the first two cords, graphically, to the scale of 1 inch = 10 pounds.

4. A 40-pound mass is supported by two cords, one of which makes an angle of 35 degrees to the right of the vertical, and the other exerts a pull of 25 pounds. Find the direction of the second cord and the tension in the first one, graphically, to the scale of 1 inch = 10 pounds. There are two solutions. Are both solutions possible with cords?

5. A 40-pound mass is supported by a cord and a rod hinged at the ends. One of these makes an angle of 24 degrees to the right of the vertical, and the other makes an angle of 85 degrees to the right of the vertical. Find the tension in the cord and the compression in the rod.

44. Equilibrium by Resolutions.—The force diagram for concurrent, coplanar forces in equilibrium is a closed polygon. If a closed polygon is projected upon any line in its plane, the sum of the positive projections is equal to the sum of the negative projections, and the total projection is zero. It follows, therefore, that when a set of forces are in equilibrium, the sum of

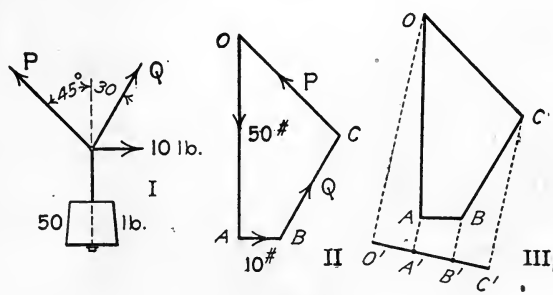


FIG. 52.

the components of these forces along any direction is zero. Fig. 52, I, is the space diagram for Problem 3 of Art. 43 and $OABCO$, in Fig. 52, II, is the force polygon. Figure 52, III, shows the projections of the sides of the force polygon upon a line at an angle of 10 degrees with the horizontal. The projections $O'A'$, $A'B'$, and $B'C'$, extend from *left* to *right*. The projection $C'O'$, extends from *right* to *left* to the point of beginning.

To solve a problem of equilibrium of concurrent forces by resolutions, it is not necessary to draw the force polygon, as the components may be calculated from the space diagram. However, *always draw the space diagram and mark all the forces which act on the free body.*

Figure 53 shows the space diagram for four forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$, which make angles $\alpha_1, \alpha_2, \alpha_3$, and α_4, respectively, with the

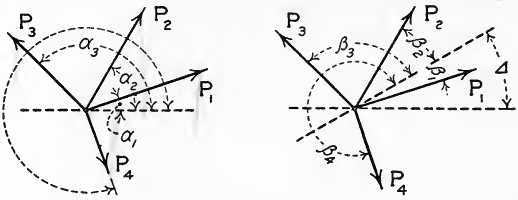


FIG. 53.

horizontal line toward the right. The point at which the forces meet is the free body. The components along the horizontal are $P_1 \cos \alpha_1$, $P_2 \cos \alpha_2$, etc. If the forces are in equilibrium,

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + P \cos_4 \alpha_4 = 0. \quad (1)$$

If the forces of Fig. 53 make angles β_1, β_2, β_3, and β_4 with a second axis, the second condition of equilibrium is

$$P_1 \cos \beta_1 + P_2 \cos \beta_2 + P_3 \cos \beta_3 + P_4 \cos \beta_4 = 0. \quad (2)$$

Any number of equations may be written by changing the direction of the axis of resolution. Since it will be shown later that only two such equations can be independent, nothing is gained by writing more than that number.

Example

A 40-pound mass is supported by three flexible cords in the same vertical plane. One cord makes an angle of 35 degrees to the left of the vertical and exerts an unknown pull of P pounds. A second cord makes an angle of 25 degrees to the right of the vertical and exerts an unknown pull of Q pounds. The third cord makes an angle of 80 degrees to the right of the vertical and exerts a pull of 10 pounds. Find the unknown forces by resolutions.

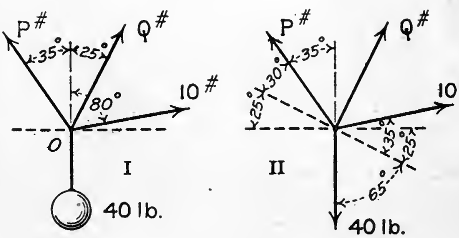


FIG. 54.

The point O, Fig. 54, I, at which all these cords meet, is the *free body* in equilibrium. Resolving horizontally and considering components toward the right positive components,

$$10 \cos 10° + Q \sin 25° - P \sin 35° = 0. \quad (3)$$

Resolving vertically and considering upward components positive,

$$10 \sin 10° + Q \cos 25° + P \cos 35° = 40. \quad (4)$$

Substituting the values of the trigonometric functions and transposing,

$$0.4226Q - 0.5736P = -9.848 \quad (5)$$
$$0.9063Q + 0.8192P = 40 - 1.736 = 38.264 \quad (6)$$

To eliminate Q multiply Equation (5) by 0.9063 and Equation (6) by 0.4226, and subtract.

$$0.4226 \times 0.9063Q - 0.5199P = -8.925,$$
$$0.4226 \times 0.9063Q + 0.3462P = 16.170,$$
$$0.8661P = 25.095,$$
$$P = 28.97 \text{ lb.}$$

Substituting the value of P in Equation (5),

$$0.4226\,Q = 28.97 \times 0.5736 - 9.848 = 16.617 - 9.848 = 6.769.$$
$$Q = 16.02 \text{ lb.}$$

This method of horizontal and vertical resolution involves two unknowns in each equation. Since the coefficients of these unknowns are sines and cosines taken to four significant figures, considerable labor is required in order to eliminate one unknown and solve the equations. It is better to make the resolutions in such a way that one equation will contain only one unknown. This may be done by resolving perpendicular to the direction of the other unknown force, since the component of a force along a line perpendicular to its direction is zero. In the above example resolve along the broken line of Fig. 54, II, which is perpendicular to the direction of **Q**. Since the force **Q** makes an angle of 25 degrees with the vertical, this line perpendicular to **Q** makes an angle of 25 degrees with the horizontal. On the right, this line makes an angle of 35 degrees with the direction of the 10-pound force and an angle of 65 degrees with the vertical line of the weight. On the left it makes an angle of 30 degrees with the direction of the force **P**. Giving the positive sign to the component toward the left, and putting the negative terms on the right of the equality sign,

$$P \cos 30° = 10 \cos 35° + 40 \sin 25°, \quad (7)$$
$$0.866\,P = 10 \times 0.8192 + 40 \times 0.4226, \quad (8)$$
$$0.866\,P = 8.192 + 16.904 = 25.096,$$
$$P = 28.98.$$

The force **Q** may now be calculated by a resolution perpendicular to the force **P**, or by substitution in Equation (5). After both **P** and **Q** are found, both may be checked by substitution in Equation (6),

$$10 \times 0.1736 = 1.736$$
$$16.03 \times 0.9063 = 14.528$$
$$28.98 \times 0.8192 = \underline{23.740}$$
$$40.004 - 40 = 0.004.$$

Problems

(*Draw the space diagram for each problem. Mark each force which acts on the free body.*)

1. A 60-pound mass is suspended by two cords, one of which makes an angle of 32 degrees to the left of the vertical, and the other makes an angle of 27 degrees to the right of the vertical. Find the tension in each cord by a resolution perpendicular to the other cord, and check by a vertical resolution. *Ans.* 31.78 lb. at 32°; 37.09 lb. at 27°.

2. A 40-pound mass is held on a smooth inclined plane, which makes an angle of 27 degrees with the horizontal, by means of a cord parallel to the plane. Find the tension in the cord by a resolution parallel to the plane, and find the normal reaction of the plane. Check both by a vertical resolution. *Ans.* 18.16 lb. tension; 35.64 lb. normal pressure.

3. A 40-pound mass is held on a smooth inclined plane, which makes an angle of 27 degrees with the horizontal, by means of a cord, which makes an angle of 40 degrees with the horizontal. Find the tension in the cord and the normal reaction and check.

Ans. 18.64 lb. tension; 31.45 lb. normal pressure.

4. A mass of m pounds is held on a smooth inclined plane, which makes an angle θ with the horizontal, by means of a cord parallel to the plane. Find the tension in the cord and the normal reaction of the plane by resolutions parallel and perpendicular to the plane, and check by a resolution vertical. *Ans.* Tension $= m \sin \theta$; normal $= m \cos \theta$.

5. A mass of m pounds is held on a smooth inclined plane, which makes an angle θ with the horizontal, by means of a horizontal cord. Find the tension in the cord and the normal reaction, and check.

Ans. Tension $= m \tan \theta$; normal $= m \sec \theta$.

6. Figure 55 shows a rod AB, 8 feet in length, which is hinged at B and supported in a horizontal position by a cord at A. The cord is fastened to a point C, which is 6 feet above the hinge B. A 60-pound mass is suspended from A. Neglecting the weight of the rod, and regarding all forces as concurrent at A, find the tension in the cord and the horizontal reaction in the rod. Solve by vertical and horizontal resolutions, and check by a resolution parallel to the cord.

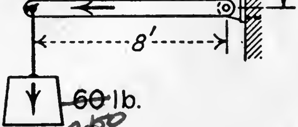


FIG. 55.

When a force is unknown, both as to direction and magnitude, it is best to consider this force as made up of two components at right angles to each other, and find the magnitude of each of these components separately. Next find the direction and magnitude of their resultant, which is the force required. This method is the same as the one for finding the resultant of a set of concurrent forces.

Example

A 50-pound mass is supported by two cords, one of which makes an angle of 75 degrees to the right of the vertical and exerts a pull of 30 pounds. Find the direction of the second cord and the tension which it exerts.

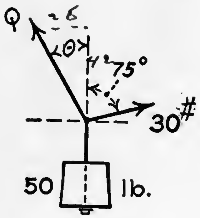


FIG. 56.

In Fig. 56, the unknown force **Q** at an unknown angle θ with the vertical may be regarded as made up of a horizontal component **H** and a vertical component **V**. Resolving horizontally,

$$H = 30 \cos 15° = 30 \times 0.9659 = 28.977.$$

Resolving vertically,

$$V = 50 - 30 \sin 15° = 50 - 30 \times 0.2588 = 50 - 7.764 = 42.236,$$

$$\text{Tan } \theta = \frac{28.977}{42.236} = 0.6861,$$

$$\theta = 34° \, 27',$$

$$Q = \frac{42.236}{\cos \theta} = \frac{42.236}{0.8246} = 51.22.$$

Check by resolving parallel to **Q**,

$$50 \cos 34° \, 27' + 30 \sin 19° \, 27' = Q,$$

$$50 \times 0.8246 = 41.23$$

$$30 \times 0.3330 = 9.99$$

$$51.22 = Q$$

Problems

7. A mass of 80 pounds is supported by three cords in the same vertical plane. One cord exerts a pull of 20 pounds in a horizontal direction toward the right. The second cord makes an angle of 35 degrees to the right of the vertical and exerts a pull of 50 pounds. Find the direction of the third cord and the tension which it exerts. Check.

8. A 50-pound mass is supported by two cords, one of which makes an angle of 38 degrees to the right of the vertical, and the other exerts a pull of 35 pounds. Find the tension in the first cord, and the direction of the second. Check the results.

Ans. 23° 35′ or 80° 25′ with the vertical; 22.74 lb. or 56.06 lb.

9. A mass of 100 pounds is supported by two cords, each of which makes an angle of 55 degrees with the vertical. Find the tension in each by two resolutions. Check.

10. Solve Problem 9 when each cord makes an angle of 80 degrees with the vertical. Check by means of the force triangle drawn to the scale of 1 inch = 40 pounds.

11. A mass of 80 pounds is supported by two cords, one of which exerts a pull of 60 pounds; the other exerts a pull of 50 pounds. Find the direction of each cord by horizontal and vertical resolutions.

Ans. The cord which pulls 60 pounds makes an angle of 38° 38′ with the vertical. The other cord makes an angle of 48° 31′ with the vertical.

45. Trigonometric Solution.—When three concurrent forces are in equilibrium, the force polygon is a triangle. It is sometimes convenient to draw this triangle and calculate the unknown forces or directions trigonometrically. This method is especially desirable when two of the forces are perpendicular to each other so that the force polygon is a right-angled triangle. It is the *best* method when the forces are expressed literally instead of numerically. When the forces form an oblique-angled triangle, and the known forces are given in numbers, the method of resolutions is better.

Example

A 40-pound mass is supported by two cords, one of which is horizontal and exerts a pull of 16 pounds. Find the direction of the second cord and the tension which it exerts.

The force triangle, Fig. 57, is a right-angled triangle of base 16 and altitude 40. Tan $\theta = 0.4$, $P = 40 \sec \theta$.

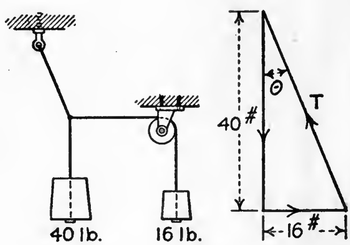


FIG. 57.

Problems

(In these problems draw the space diagram and the force diagram. Unless it is desired to check by means of the graphical solution, it is not necessary that these diagrams be drawn to scale. The force diagram must be drawn separate from the space diagram.)

1. Solve Problem 2 of Art. 44 by the trigonometric solution of the force triangle.

2. Solve Problem 6 of Art. 44 by means of the force triangle.

3. Solve Problem 1 of Art. 44 by means of the oblique-angled force triangle. Use the law of sines. Is the method shorter than that of two resolutions?

4. A mass of m pounds is held on a smooth inclined plane, which makes an angle α with the horizontal, by a rope which makes an angle β with the horizontal. Find the tension in the rope and the normal reaction by means of the solution of the force triangle.

Ans. Tension $= \dfrac{m \sin \alpha}{\cos (\alpha - \beta)}$; normal $= \dfrac{m \cos \beta}{\cos (\alpha - \beta)}$

It sometimes happens that the force triangle is similar to a triangle of the space diagram. In that case the magnitudes of the forces are proportional to the corresponding lengths in the space diagram.

Example

Figure 58 shows a horizontal rod, 12 feet in length, which is hinged at the left end A. A cord is attached to the right end B and fastened to a point C, which is 8 feet above A. Neglecting the weight of the rod, find the tension in the cord and the compression in the rod when a load of 40 pounds is placed at B.

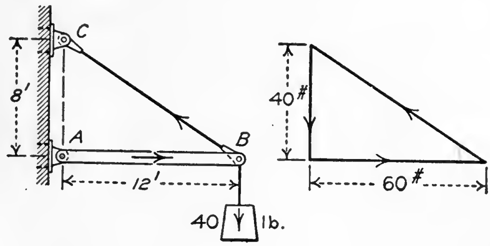


FIG. 58.

The force of 40 pounds in the force diagram is homologous to the length of 8 feet in the space diagram. The ratio is 5 to 1. The horizontal push in the rod is $5 \times 12 = 60$ pounds. In the space diagram the length BC is $\sqrt{208} = 14.422$ feet. The tension in the cord is $5 \times 14.422 = 72.11$ pounds.

Problems

5. In Fig. 58, the cord BC is shortened until the point B is 2 feet higher than A. Find the compression in the bar and the tension in the cord.

Ans. 60 lb.; 66.33 lb.

6. If the length AB of Fig. 58 is constant and the point C is at a constant distance *directly above* A, show that the compression in the rod will be the same for any position of AB.

46. Number of Unknowns.—In each of the problems of the equilibrium of concurrent, coplanar forces in the preceding articles there have been two unknown quantities. These have been:

(1) An unknown magnitude and an unknown direction of one force;

(2) An unknown magnitude of one force and an unknown direction of another;

(3) Two unknown magnitudes;

(4) Two unknown directions.

With two unknowns, two independent equations are required to solve each problem.

As far as the mechanics of the problem is concerned, there may be only two unknowns in a problem of concurrent, coplanar forces in equilibrium. This may be shown in several ways. If the problem is solved by resolutions, only two equations can be written which are independent of each other. Fig. 59 shows a force AB. Its orthographic component along a direction OC is the length A_1B_1. Its orthographic component along another direction OD is the length A_2B_2. If a third orthographic component were drawn, its length could be expressed in terms of A_1B_1, A_2B_2, and the angles. Consequently, if three equations were written for a problem of concurrent, coplanar forces, any one of these equations could be derived algebraically from the other two, and would not be independent. Since there can be only two independent equations for a problem of concurrent, coplanar forces, it is evident that there may be only two unknowns, unless conditions are given which are not based upon mechanics.

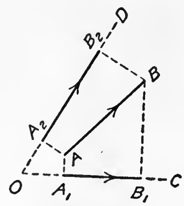


FIG. 59.

The number of possible unknowns may be determined in a different way from the geometry of the force polygon. All the known forces may be represented in the force diagram by the single vector of their resultant. The problem of equilibrium involves the solution of a polygon of which this resultant vector is one side. This polygon is a triangle for all problems which may be solved from mechanical considerations. A triangle may be solved in the following cases:

1. One side and two angles given, with the *length* of *two sides* unknown.

2. Two sides and the included angle given, with the *magnitude* and *direction* of the third side unknown.

3. One side given together with the length of a second side and the direction of the third side. The *length* of one side and the *direction* of another are unknown.

4. The length of all sides given, with two directions unknown.

Some of these cases apparently have three unknowns. When two angles of a triangle are given, the third angle is regarded as known.

It is evident from the foregoing statements that a problem of the equilibrium of concurrent, coplanar forces may have only two unknown quantities. Sometimes it happens that there is a greater number of unknowns. Such a problem can not be solved completely unless additional conditions are given.

Example

A body is in equilibrium under the action of three forces. These are: a force of 40 pounds horizontally toward the right, a force **P** at an angle of 25 degrees to the left of the vertical, and a force **Q** of unknown direction and magnitude.

There are three unknowns, and an attempt at a graphical solution, Fig. 60, shows that it may be solved in an infinite number of ways. Another condition must be added to make the problem definite. Let the force **Q** of unknown direction be twice as great as the force **P** at 25 degrees to the left of the vertical. This is an additional algebraic condition which does not depend upon mechanics.

$$Q = 2P \tag{1}$$

Resolving horizontally and vertically,

$$P \sin 25° + 2P \sin \theta = 40, \tag{2}$$

$$P \cos 25° = 2P \cos \theta, \tag{3}$$

$$\cos \theta = \frac{\cos 25°}{2} = \frac{0.9063}{2} = 0.4531.$$

Problems

1. In Fig. 60, let the force **Q** be three times the force **P**. Find the magnitude of each force and the direction of **Q**.

Ans. $P = 12.19$ lb.; $Q = 36.57$ lb.

2. Solve Problem 1 graphically.

3. A body is in equilibrium under the action of four forces. These are: a force of 40 pounds at 0 degrees, a force **Q** at 110 degrees, a force **P** at 245 degrees, and a force equal to Q at 180 degrees. Solve graphically.

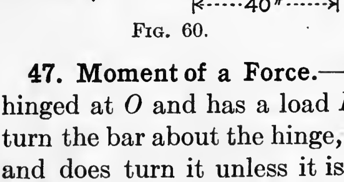


FIG. 60.

47. Moment of a Force.—Figure 61 shows a bar OB, which is hinged at O and has a load P applied at B. The load tends to turn the bar about the hinge, and does turn it unless it is balanced by a second force tending to turn in the opposite direction. In Fig. 61, I, the turning effect of the force is greater than in the position of Fig. 61, II. In the position of Fig. 61, III, the force has no tendency to turn the bar about the hinge.

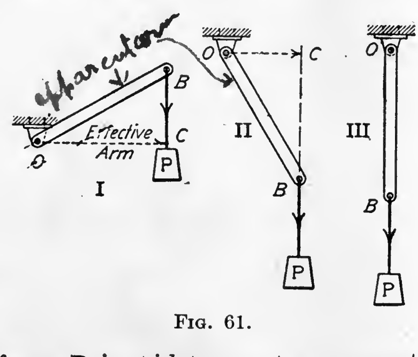


FIG. 61.

In the other positions, the force **P** is said to exert a moment on the bar about the hinge. The point about which moment is taken is called the *origin* of *moments* or the *center* of *moments*.

Definition.—The moment of a force about a point in its plane is the product of the magnitude of the force multiplied by the perpendicular distance from the point to the line along which the force acts.

In Fig. 61, the moment of the force **P** is the product of the force multiplied by the length *OC*. If the force is measured in pounds and the distance in feet, the moment is expressed in foot-pounds. To distinguish moment from work, some writers use pound-feet for moment and foot-pounds for work. Though this distinction is desirable, it has not come into general use.

The perpendicular distance from the origin of moments to the line of action of the force is the *effective moment arm*. The length *OB*, from the origin of moments to the point of application of the force, may be called the *apparent moment arm*. If the length of the apparent moment arm is *a*, Fig. 62, and the angle between its direction and the direction of the force is α,

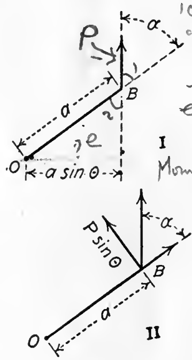


FIG. 62.

effective arm $= a \sin \alpha$ (1)

moment $= P \times a \sin \alpha$. Formula IV

Counter-clockwise moment is generally regarded as positive, especially in works of mathematical nature; and clockwise moment is regarded as negative. In algebraic equations, moment will be represented in this book, by *M*.

Problems

1. A force of 60 pounds, at an angle of 25 degrees to the right of the vertical, is applied to the right end of a bar, which is 3 feet in length. The bar makes an angle of 80 degrees to the right of the vertical. Find the moment of this force about the left end of the bar.

Ans. $M = 147.45$ foot-pounds.

2. A force of 12 pounds, at an angle of 20 degrees to the left of the vertical, is applied to a point whose coördinates are (4, 3). Find the moment of this force about the origin of coördinates.

Ans. $M = 57.42$ foot-pounds.

3. Solve Problem 2 if the force is at an angle of 20 degrees to the right of the vertical.

By changing the order of the letters of Formula IV, it may be written,

$$M = P \sin \alpha \times a. \tag{2}$$

The term $P \sin \alpha$ is the component of the force perpendicular to the direction of the apparent moment arm. Equation (2) gives this second definition of moment: *The moment of a force is the product of the apparent arm multiplied by the component of the force which is perpendicular to it.*

The first definition states that moment is the entire force multiplied by the component of the arm; the second definition states that moment is the product of the entire arm multiplied by the component of the force.

Since force and apparent arm are vectors, moment is the product of two vectors. It will be shown in Art. 108 that this product is a vector.

48. Moment of Resultant.—If P_1, P_2, P_3, etc. are forces which make angles α_1, α_2, α_3, etc. with the apparent arm of length a, the sum of the moments of all these forces about the origin is

$M = P_1 \sin \alpha_1 a + P_2 \sin \alpha_2 a + P_3 \sin \alpha_3 a = (P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3)a$. The term $P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3 +$ etc. in the above equation is the component of the resultant force perpendicular to the apparent arm; consequently, *the moment of the resultant of several concurrent forces is equal to the sum of the moments of separate forces.*

When the moment of a force is calculated by means of the component perpendicular to the apparent arm, it is really calculated by means of the moment of two components at right angles to each other. From Fig. 62, II, the second component is $P \cos \alpha$ in the direction of the apparent arm. Since the line of action of this component passes through the origin of moments, its moment is zero. The total moment is, then, the moment of the component normal to the apparent arm.

It is often advisable to resolve both the force and the apparent arm into components. This method is especially convenient when the coördinates of the ends of the apparent arm are given instead of its length and direction.

Example

A force of 40 pounds at an angle of 25 degrees to the left of the vertical is applied at the point whose coördinates are (6, 2). Find the moment of this force about the origin of coördinates.

First method.

The apparent moment arm is *OB*, Fig. 63.

$$OB = 6.325 \text{ ft.}$$

$$\text{Tan } \theta = \frac{1}{3}\ ;\ \theta = 18° \ 26'.$$

$$OC = OB \cos (25° - \theta) = OB \cos 6° \ 34' = 6{,}325 \times 0.9934 = 6.283 \text{ ft.}$$

$$M = 40 \times 6.283 = 251.32 \text{ foot-pounds.}$$

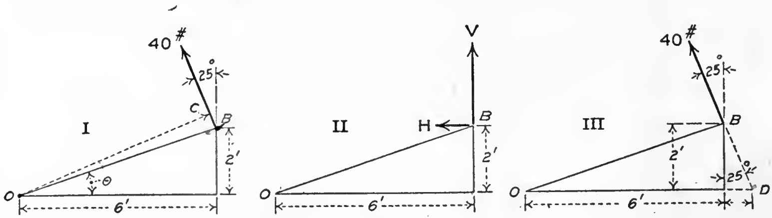


FIG. 63.

Second method.

Resolving horizontally, Fig. 63, II,

$$H = 40 \times 0.4226 = 16.904.$$

Resolving vertically,

$$V = 40 \times 0.9063 = 36.252.$$

The effective arm of the horizontal component is the vertical length of 2 feet. The effective arm of the vertical component is the horizontal distance of 6 feet. Both components turn counter-clockwise.

$$\begin{array}{rl} 16.904 \times 2 = & 33.808 \\ 36.252 \times 6 = & \underline{217.512} \\ & 251.320 \text{ foot-pounds.} \end{array}$$

Problems

1. Solve the example above if the force makes an angle of 25 degrees to the right of the vertical.

2. A force of 50 pounds at an angle of 15 degrees to the right of the vertical is applied at the point (5, 4). Find the moment of this force about the origin of coördinates, and also about the point (2, 2).

A force may be regarded as applied at *any* point along its line of action. The effective moment arm in Fig. 63, III, is the same, whether the force is applied at *B* or *D*, or at any other point along its line. If the force of 40 pounds be regarded as applied at *D*, on the axis of *X*, the moment about *O* is the product of the length *OD* multiplied by the vertical component of the force.

Problems

3. Calculate the length OD of Fig. 63, III, trigonometrically, and multiply by the vertical component of the 40-pound force.

4. A force of 50 pounds at an angle of 40 degrees to the left of the vertical passes through the point (4, 4). Find its moment about the origin of coördinates. Solve by both methods of the example above. Also find the intersection of its line of action with the X axis, and multiply the abscissa of this point by the vertical component of the force. Find the intersection with the Y axis and multiply the Y-intercept by the horizontal component.

5. A force of 10 pounds at an angle of 35 degrees to the left of the vertical passes through the point (6, 5). Find its moment with respect to the point (2, 2). Solve by means of the horizontal and vertical components. Solve by the vertical component alone. Solve by the horizontal component alone. Find the distance of the point (2, 2) from the line of action of the force.

6. Find the distance of the point (3, 5) from a line through the point (6, 7) at an angle of 20 degrees to the left of the vertical. Solve by principles of moments.

7. A force of 20 pounds is applied along a line which passes through the points (0, 6) and (8, 0). Find its moment about the origin of coördinates. Solve by the vertical component with the horizontal arm. Check by the horizontal component with the vertical arm.

49. Equilibrium by Moments.—Since the moment of the resultant of a set of concurrent forces is equal to the sum of the moments of the separate forces, and, since the resultant of a set of forces in equilibrium is zero, it follows, that *the sum of the moments of a set of concurrent forces about any origin is zero when the forces are in equilibrium.* This affords another method of solving a problem of the equilibrium of concurrent, coplanar forces. The moment equation of equilibrium is

$$M = P_1 \sin \alpha_1 \, a + P_2 \sin \alpha_2 \, a + P_3 \sin \alpha_3 \, a + \text{etc.} = 0. \quad (1)$$

$$M = (P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3 + \text{etc.}) \, a = 0. \quad (2)$$

The term in the parenthesis in Equation (2) is the sum of the components, perpendicular to the apparent moment arm of length a, of all the forces which act on the free body. It is evident from this equation, after it has been divided by the apparent arm, that a moment equation for concurrent forces is equivalent to a resolution perpendicular to the apparent arm.

In the solution of a problem of concurrent forces, a moment equation is used, instead of a resolution equation, when it is easier to calculate the length of the effective arm than it is to resolve the force into its components. In a structure or machine the effective force may often be measured directly. Figure 64 represents a bar hinged at O and supporting a load **P** at B. The bar is supported by a cord at B. It is desired to find the tension in this cord. The three forces at B are the load **P**, the tension in the cord **T**, and the compression along the bar. If moments are taken about the hinge O, the line of action of the compression in the bar passes through this point so that its moment is zero. The moment of **T** in a counter-clockwise direction must be equal to the moment of **P** in a clockwise direction. The effective moment arm of **T** is the length OD, measured from O perpendicular to the direction of the cord. The moment arm of **P** is the perpendicular distance from the hinge to its line of action—the length OC of Fig. 64. To find the perpendicular distance from a point to a straight line, it is only necessary to measure the shortest distance. This may be done with a tape-line or rule. In the actual machine or structure, it is much easier to measure these lengths than it is to determine the angles necessary for resolution equations.

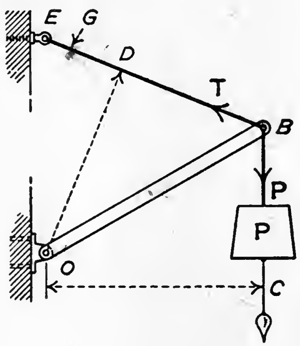


FIG. 64.

The moment equation for Fig. 64 is

$$T \times OD = P \times OC \tag{3}$$

To find the compression in the bar of Fig. 64, moments may be taken about some point in the line of the cord. The point G may well be used as the center of moments.

Problems

1. A mass of 200 pounds is suspended by two cords, which are fastened to the mass at a point A. One cord is attached to a fixed point, which is 5 feet higher than A and 8 feet to the left of the vertical line through it. The second cord is attached to another point, which is 7 feet higher than A and 4 feet to the right of the vertical line through it. Draw the space diagram to scale. Find the tension in the left cord by moments about some point in the line of the right cord, measuring the moment arms from the drawing. Then find the tension in the right cord by moments about some point in the line of the left cord. Check by moments about a point directly over the load.

2. A 40-pound mass is supported by two cords, one of which is horizontal and the other makes an angle of 23 degrees with the vertical. Find the tension in the horizontal cord by moments about some point in the line of the other cord, computing the effective arm trigonometrically. Find the tension in the other cord by resolutions.

3. In an arrangement similar to Fig. 64 the bar is weightless, is 6 feet long, and makes an angle of 25 degrees with the horizontal. The cord is attached to a fixed point *E* which is 5 feet above the hinge *O*. Draw the space diagram to the scale of 1 inch = 1 foot. With the hinge as the origin of moments, measure the effective arms on the space diagram and calculate **T** when the load **P** is 60 pounds. Find the compression in the bar with the origin of moments at *G*, at a distance of 1 foot from *E*. Also solve for the compression with the origin of moments at *E*.

4. Solve Problem 3 by moments, calculating the arms instead of measuring them.

5. Solve Problem 6 of Art. 44 by moments.

50. Conditions for Independent Equations.—In a problem of concurrent, coplanar forces there are two independent equations. These may be:

(1) Two resolution equations;
(2) Two moment equations;
(3) One resolution equation and one moment equation.

A moment equation is equivalent to a resolution perpendicular to the line joining the origin of moments to the point of application of the forces. If two moment equations are written, and the line joining the point of application of the forces with one origin of moments passes through the other origin of moments, the equations will not be independent. Each moment equation will be equivalent to a resolution perpendicular to this line. After the equations have been divided by the lengths of the apparent moment arms, they are identical. *When two moment equations are written for a problem of concurrent, coplanar forces, the two origins of moment and the point of application of the forces must not lie in the same straight line. When one resolution and one moment equation are written, the resolution must not be perpendicular to the line joining the origin of moments with the point of application of the forces.*

A resolution perpendicular to the direction of an unknown force eliminates that force. A moment about a point in the line of action of an unknown force eliminates that force.

51. Connected Bodies.—It frequently happens that two points, each of which is subjected to a set of concurrent forces, are connected together by a single member, such as a cord or hinged rod.

Each point is in equilibrium under the action of the external forces and of the force in the connecting member.

In Fig. 65, the forces are concurrent at A and B. The member which joins A to B exerts a downward pull **T** on B and an equal upward pull on A. Including the direction and magnitude of the unknown tension **T**, there are four unknowns at B. It is not possible to solve for these unknowns by equilibrium equations at B, unless the number of unknowns is reduced to two. At A, on the other hand, there are only two unknowns, and the problem may be solved. *In problems of this kind, begin at a point where there are only two unknowns.*

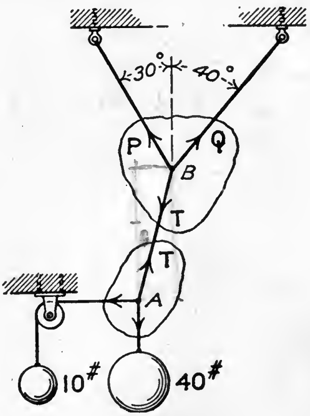


FIG. 65.

The equilibrium at A may be solved by one moment and one resolution equation. If the length AB is a and the angle with the vertical is θ, moments about B give,

$$10\,a \cos\theta = 40\,a \sin\theta, \qquad (1)$$
$$\tan\theta = 0.25;\ \theta = 14°\,02'.$$

A vertical resolution gives,

$$T\cos\theta = 40;\ T = 40 \sec\theta = 40 \times 1.0307 = 41.23 \text{ lb.}$$

The force **T** may be checked from the force triangle by means of the square root of the sum of the squares of 10 and 40. This method is convenient when the numbers are small. The trigonometric method is better for large numbers.

A closed curve has been drawn about the point A of Fig. 65. This curve encloses the joint, which is the free body. The arrows inside the curve represent the direction of the forces which act on the free body. The arrow in the member AB points upward, which shows that AB pulls upward on the joint. Since AB pulls upward on the joint at A, the joint pulls downward on AB. The stress in AB is tensile.

At joint B the tension in AB pulls downward. The direction and magnitude of this tension are known. Its components are 10 pounds horizontally toward the left and 40 pounds vertically downward. The force **T** at A is the *equilibrant* of the forces of

10 pounds and 40 pounds. The force **T** at B is the *resultant* of these two forces. The unknowns at B are now the forces **P** and **Q** of known direction. The free body is now the joint B. Resolving perpendicular to **Q** and using the components of **T** instead of the resultant force,

$$P \sin 70° + 10 \cos 40° = 40 \sin 40° \qquad (2)$$

$$\begin{aligned} 40 \times 0.6428 &= 25.712 \\ -10 \times 0.7660 &= -7.660 \\ \hline 0.9397\,P &= 18.052 \\ P &= 19.20 \text{ lb.} \end{aligned}$$

Resolving vertically,

$$Q \cos 40° + P \cos 30° = 40 \qquad (3)$$

$$0.7660Q = 40 - 19.20 \times 0.8661 = 40 - 16.64 = 23.36$$

$$Q = 30.49 \text{ lb.}$$

These results may be checked by horizontal resolution.

Problems

1. In Fig. 66, find the unknown forces at the right end by one moment and one resolution equation or by two resolution equations. Put the arrows showing the direction of the forces **R** and **Q** inside the closed curve.

Ans. **R** = 160 lb. compression; **Q** = 200 lb. tension.

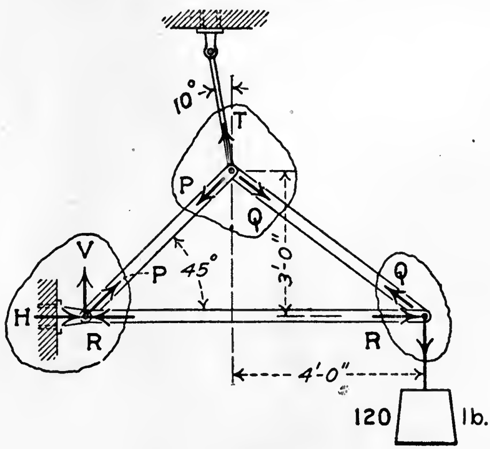


FIG. 66.

2. Using the value of **Q** found in Problem 1 (or its components) find the magnitude of the forces **P** and **T** at the top. Draw the arrows inside the closed curve at this point to show the forces acting on the joint.

Ans. **T** = 241.7 lb. tension; **P** = 166.9 lb. tension; horizontal component of **P** = 118.02 lb.

3. At the left end, put arrows in the closed curve to give the direction of the forces **P** and **R**. Find the horizontal and vertical components of the hinge reaction **S**, and find the direction and magnitude of **S**.

Ans. **H** = 41.98 lb. toward the right; **V** = 118.02 lb. downward.
S = 125.3 lb. at 19° 35′ with the vertical.

4. Solve Problems 1, 2, and 3 if the force **T** is vertical.

5. Solve Problems 1, 2, and 3 if the force **T** makes an angle of 10 degrees to the right of the vertical.

52. Bow's Method.—The forces in a connected system are often determined graphically. It is possible to begin at some point at which there are only two unknowns and draw the force polygon. The force in the member which joins this point with a second point may then be used in the force polygon for that point. Instead of drawing two lines for the force in the connecting member, and making two separate force polygons, it is better to draw this common line but once, and to connect the two polygons into a single diagram. To avoid confusion in complicated diagrams, a system of lettering the space diagram and force diagram has been adopted by many workers in graphical statics. This is called, from the name of the inventor, Bow's method. Each letter in the space diagram represents an area bounded by members which transmit force in the direction of their length. The force in a given member is represented by the two letters which the member separates in the space diagram. In the force diagram, one of these letters is put at each end of the line representing the force. In this book, Italic capitals will be used on the space diagram and lower case Italics on the force diagram.

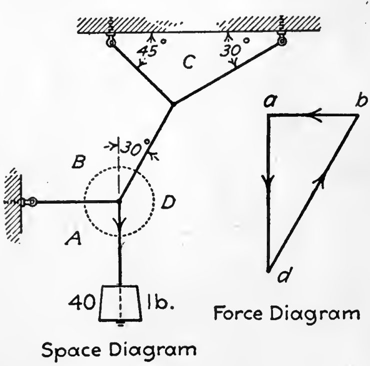


FIG. 67.

In Fig. 67, the letter *A*, on the space diagram, represents the area to the left of the vertical cord and below the horizontal cord. The letter *B* represents the area above the horizontal cord and to the left of the cord at 30 degrees with the vertical. The horizontal cord in the space diagram separates the area *A* from the area *B*. The line from *a* to *b* in the force diagram represents the force in the horizontal cord.

To solve the problem graphically, the space diagram is first drawn to scale and the spaces are lettered. The direction and magnitude of the force in the vertical cord are known. To represent the direction, an arrow pointing downward is placed on this member in the space diagram. The force diagram is begun by drawing a vertical line 40 units in length. Since the vertical cord separates area *A* from area *D*, one end of this line in the force diagram is marked *a* and the other end is marked *d*. In Fig. 67, the letter *a* is put at the top. It would be just as correct to put *d* at the top. An arrow is put in the force diagram to show the direction of the force in *ad*. Since the horizontal cord separates area *A* from area *B*, the force in this member is *ab*. A line of indefinite length is drawn horizontally through *a* in the force diagram. The third force at this joint is *bd*. Through the point *d* of the force diagram, a line is now drawn parallel to the cord which separates *B* from *D* in the space diagram. The intersection of this line through *d* with the horizontal line through *a* gives the point *b*. The arrows are now placed in the force diagram. Since the arrow in *ad* points *toward d* the arrow in *db* must point *from d* toward *b* and the arrow in *ba* from *b* toward *a*. Corresponding arrows are now drawn at the lower joint in the space diagram. These is shown in Fig. 68.

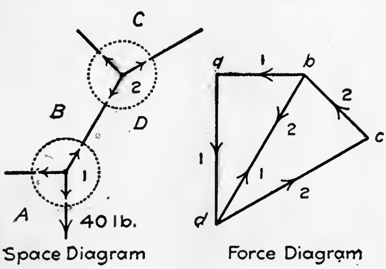


FIG. 68.

Figure 68 shows the final diagrams for both joints, while Fig. 67 gives the first steps in the construction of these diagrams. To avoid confusion, the joints are numbered 1 and 2. The arrows which show the directions are marked with similar numerals in the force diagram. The arrows marked 1 in Fig. 68 are the same as those in the force diagram of Fig. 67.

Since the force *bd* at the lower joint is upward, it must be downward at the upper joint. An arrow pointing downward is now placed on this member at the upper joint, as shown in Fig. 68. In the line *bd* of the force diagram a similar arrow is placed and marked with the numeral 2. A line is then drawn through *b* parallel to the member *BC* of the space diagram, and a line is drawn through *d* parallel to *DC* of the space diagram. The intersection of these lines gives the point *c*. Since the arrow marked 2 in the force diagram is from *b* toward *d*, the arrow in *dc* must point from *d* toward *c*, and the arrow in *cb* must point from *c* toward *b*. Finally, the corresponding arrows are drawn at joint 2 of the space diagram.

The length of the lines on the final force diagram are now measured, and the results expressed in pounds. These figures are frequently written in the space diagram. All the stresses in Fig. 68 are tensile. In most problems, some stresses are tensile and some are compressive. The character of the stress is marked on the space diagram.

It is understood of course, that Fig. 67 is merely one step of Fig. 68, drawn separately for clearness of explanation. In solving the problems below, draw a single space diagram and a single force diagram, as in Fig. 68. The numbered arrows are not generally used. They afford, however, great help to beginners. It is best to draw the space diagram accurately to scale. The directions on the force diagram are found by drawing lines parallel to the members of the space diagram.

Problems

1. Solve the example of Fig. 67 graphically to the scale of 1 inch = 10 pounds, marking *d* at the top and *a* at the bottom of the first vertical line. Compare the final force diagram with that of Fig. 68.

2. Solve the example of Fig. 66 graphically to the scale of 1 inch = 1 foot

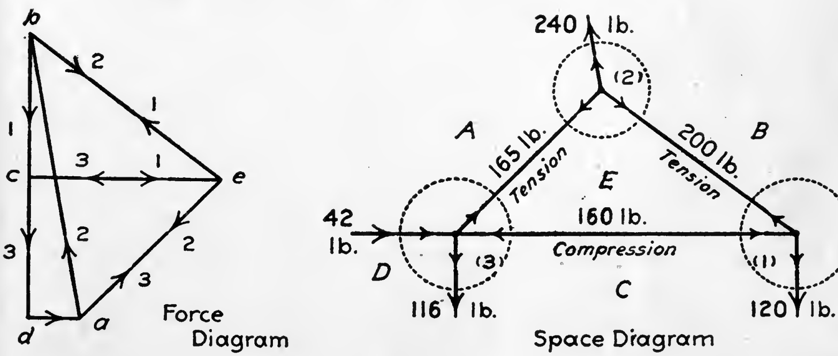


FIG. 69.

in the space diagram, and 1 inch = 40 pounds in the force diagram. Scale the force diagram and put the results in pounds on the space diagram, as in Fig. 69.

NOTE.—The values of Fig. 69 were taken from the original drawing of the force diagram, which has been reduced in scale for printing.

3. The mast of a derrick is 15 feet long and the boom is 20 feet long. One guy rope, in the same vertical plane as the boom, makes an angle of 30 degrees with the horizontal. The boom makes an angle of 15 degrees with the horizontal and carries a load of 300 pounds. Find all internal forces, the tension in the guy rope, and the horizontal and vertical components of the reaction at the bottom ofthe mast. Use the scale of 1 inch = 5 feet on the space diagram, and 1 inch = 100 pounds on the force diagram.

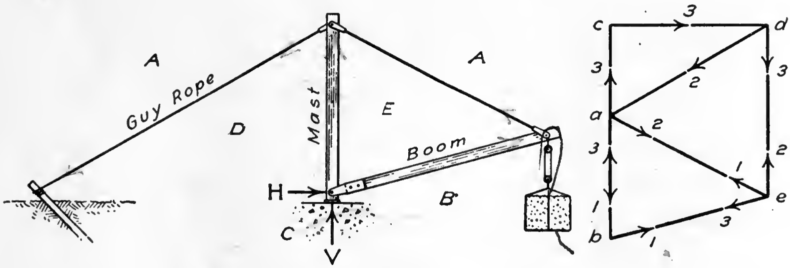


FIG. 70.

NOTE.—In an actual derrick, the boom is attached to the mast at some distance above the bottom. The cable which supports the load runs parallel to the boom, and increases the compression in that member. The boom is lifted by several cables, which run over pulleys. The part of the cable which comes down the mast increases the compression in that member.

4. Solve Problem 3 by moments and resolutions, beginning with the right end of the boom as the first free body. Show that the compression in the boom is the same, no matter what angle it makes with the horizontal.

(In the solution by moments and resolutions, it is convenient to represent each force by a single letter, as H, V, P, etc., instead of by two letters as in Bow's method.)

5. Solve Problem 3 graphically when the boom is elevated to make an angle of 45 degrees with the horizontal.

53. Summary.—The resultant of concurrent, coplanar forces is their vector sum.

The graphical condition of equilibrium for a set of concurrent, coplanar forces, is that the force diagram is a closed polygon.

A problem of the equilibrium of concurrent, coplanar forces may be solved by any one of the following methods:

1. *Construct the force polygon and determine the magnitude and direction of each of the unknown forces by measurement.*

2. *Construct the force polygon and solve trigonometrically.* This method is convenient when the force polygon is a triangle, especially if it is a right-angled or an isosceles triangle.

3. *Write two resolution equations.* The *sum of the components along any direction is zero.*

4. *Write two moment equations. The sum of the moments about any point is zero.* The two origins of moment and the point of application of the forces must not lie in the same straight line.

5. *Write one moment equation and one resolution equation.* The resolution must not be perpendicular to the straight line which joins the origin of moments and the point of application of the forces.

Jointed frames, in which the forces are parallel to the members connecting the joints, may be solved as a series of problems of concurrent forces. Begin with a joint at which there are only two unknowns, and solve by any of the methods above. By the use of Bow's method of lettering, a problem of this kind may be solved graphically, with all the forces on a single diagram.

A resolution perpendicular to the direction of an unknown force eliminates that force. Moment about a point in the line of action of an unknown force eliminates that force.

The moment of a force is the product of the *magnitude* of the force multiplied by the *component* of the *apparent arm perpendicular* to it; or the product of the *entire apparent arm* multiplied by the *component* of the force *perpendicular* to it. When the coördinates of the ends of the apparent arm are given, instead of its length and direction, it is advisable to resolve both the force and the apparent arm into their components.

$$M = Vx - Hy, \qquad \text{Formula V.}$$

in which V and H are the components of the force and x and y are the components of the arm. The component V is parallel to y and the component H is parallel to x.

54. Miscellaneous Problems

1. Find the direction and magnitude of the resultant of 24 pounds at 20 degrees, 30 pounds at 50 degrees, 20 pounds at 110 degrees, and 16 pounds at 210 degrees. Solve by resolutions along one pair of axes at right angles to each other. Check by resolutions along a second pair of axes.

2. A mass of 50 pounds on a smooth inclined plane, which makes an unknown angle with the horizontal, is held by a force of 18 pounds parallel to the plane. Find the inclination of the plane and the normal reaction.

3. An unknown mass is placed on a smooth inclined plane which makes an unknown angle with the horizontal. It is held in equilibrium by a force of 18 pounds parallel to the plane, or by a force of 20 pounds at an angle of 10 degrees above the horizontal. Find the mass of the body and the inclination of the plane. Also find the normal reaction of the plane in each case.

4. Solve Problem 3 graphically.

5. A 40-pound mass slides on a smooth straight rod which makes an angle of 25 degrees with the horizontal. It is attached by means of a weightless cord to a mass of 20 pounds which slides on a second smooth rod. This rod makes an angle of 35 degrees with the horizontal. The two rods are in the same vertical plane and on opposite sides of the vertical. Find the direction of the cord, the tension which it exerts, and the normal reaction of each rod.

Figure 71, II, shows the masses supported by two cords. The problem is the same as that of Fig. 71, I, and the diagram is more convenient for a graphical solution.

Considering the 40-pound mass as the free body in equilibrium, and resolving horizontally,

$$P \sin 25° = Q \cos \theta. \qquad (1)$$

Considering the 20-pound mass as the free-body in equilibrium, and resolving horizontally,

$$Q \sin 35° = T \cos \theta. \qquad (2)$$

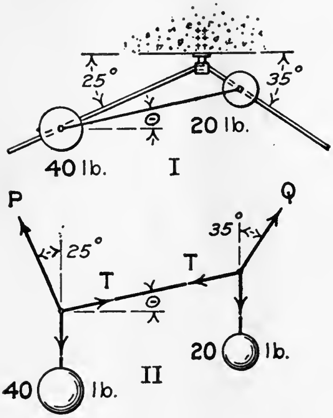


FIG. 71.

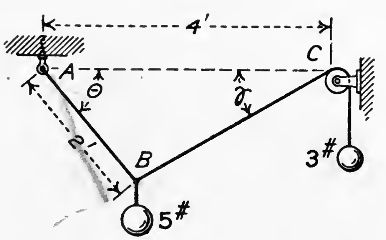


FIG. 72.

Combining Equations (1) and (2)

$$P \sin 25° = Q \sin 35°. \qquad (3)$$

Equation (3) is the same as if the forces **P** and **Q** were concurrent.

Resolving vertically,

$$P \cos 25° = 40 - T \sin \theta, \qquad (4)$$
$$Q \cos 35° = 20 + T \sin \theta, \qquad (5)$$

from which

$$P \cos 25° + Q \cos 35° = 60. \qquad (6)$$

Equation (6) is the same as if all the external forces were concurrent.

Combining Equations (3) and (6), **P** = 39.74 lb.; **Q** = 29.28 lb. Combining Equations (1) and (4), $\theta = 13° 23'$, $T = 17.23$ lb.

6. Letter Fig. 71, II, by Bow's method and solve graphically.

7. A mass of 5 pounds is supported by a cord AB, which is 2 feet long and is fixed at A, and by a second cord which runs over a smooth pulley and supports a mass of 3 pounds. The point (C of Fig. 72) at which the second cord is tangent to the pulley is 4 feet from A in a horizontal direction. Find the angle which AB makes with the horizontal when the system is in equilibrium.

Taking moments about A and eliminating trigonometrically, the equation is found to be

$$\cos^3 \theta - 1.61 \cos^2 \theta + 0.36 = 0$$

This equation may be solved by the trigonometric method for the solution of a cubic, or by the method of trial and error (see Art. 252). One root is $\cos \theta = 0.5958$. To get the remaining two roots of the cubic, divide the equation by $\cos \theta - 0.5988$, and solve the quadratic thus obtained.

Two of these roots are solutions of the problem of mechanics. (Construct the space diagram for one of these.) The third root is a solution of the mathematical equation but is impossible mechanically.

8. A 20-pound mass is supported by a cord AB which is 3 feet long, and a second cord which runs over a smooth pulley 2 feet in diameter and supports a mass of 15 pounds. The axis of the pulley is 6 feet to the right and

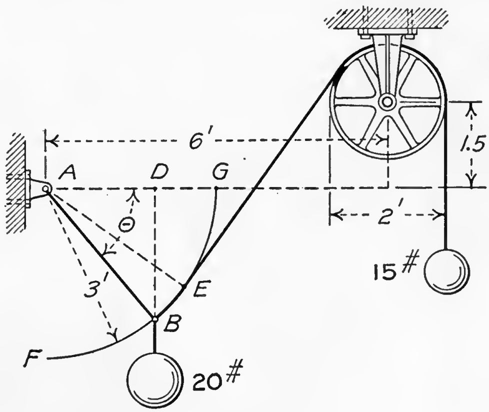


FIG. 73.

1.5 feet above the point A, the upper end of the first cord. Find the position of equilibrium.

The algebraic solution of this equation is difficult and involves the solution of an equation of higher degree by the method of trial and error. It is most easily solved by a combination of moments and graphics. With A as the center and with a radius of 3 feet, draw the arc FG, Fig. 73. Select some point B on this arc and draw AB. From B draw a line tangent to the pulley. The length AE drawn perpendicular to this tangent is the moment arm of the 15-pound force. The moment arm of the 20-pound force is AD. For equilibrium.

$$20AD = 15AE.$$

If the moment of the 20-pound force is too great, the angle θ is too small. Choose another point on the arc and repeat the process. Interpolate for the final result.

CHAPTER V

NON-CONCURRENT CO-PLANAR FORCES

55. Resultant of Parallel Forces in the Same Direction. Figure 74 shows a common case of parallel forces in the same direction. The rigid beam *ABC* carries two loads, **P** and **Q,** which are parallel forces. The *equilibrant* of these two forces is the force **S** at *B*. The *resultant* of **P** and **Q** is a force at *B*, which is equal and opposite to the equilibrant. The direction, magnitude,

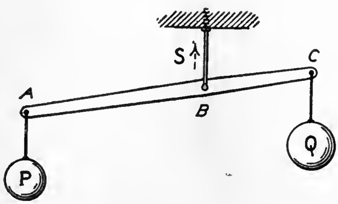


FIG. 74.

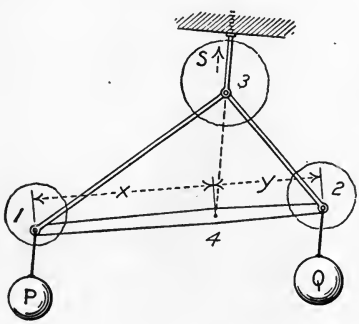


FIG. 75.

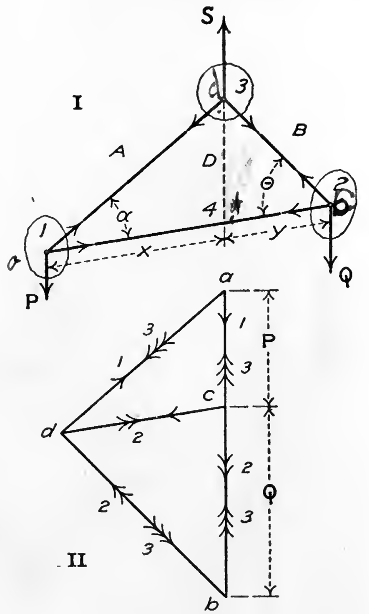


FIG. 76.

and position of the resultant may be found by first finding the equilibrant.

To find the equilibrant, replace the rigid beam *ABC* of Fig. 74 by the jointed frame of Fig. 75. Neglecting the weight of the frame, the forces at each joint may be found by the methods for concurrent forces. Figure 76, I, is the space diagram for Fig. 75. For simplicity, each member is represented by a single line. Figure

76, II, shows the force diagram constructed by Bow's method. The angle α is assumed to be known and the angle θ is assumed to be unknown. Beginning at joint No. 1, construct the force triangle *acd*. At joint No. 2, the direction of *dc* is from left toward right, and the magnitude of *cb* is known. The force triangle is *cbd*. The direction of *bd* gives the unknown angle θ. At joint No. 3, the direction and magnitude of the forces *ad* and *db* are known. The line *ba* represents the equilibrant desired.

The force *ba* of triangle No. 3 is along the same line as the forces **P** and **Q**, and is equal to their sum. *The equilibrant of two parallel forces is equal to the sum of the two forces, and is in the opposite direction.*

$$P + Q + S = 0; P + Q = -S. \qquad (1)$$

The resultant, which is equal and opposite the equilibrant, is the algebraic sum of **P** and **Q** and has the same sign.

$$P + Q = R \qquad (2)$$

The force polygon of Fig. 76, II, gives the *direction* and *magnitude* of the equilibrant, and, consequently, of the resultant. There remains the problem of finding algebraically the *position* of the equilibrant or resultant. This position is given graphically by the space diagram of Fig. 76, I. It is now desirable to express this position in algebraic language. Extend the line **S** of Fig. 76, I, till it intersects the line 1–2 at the point 4. Let the distance from point 1 to point 4 be represented by x, the distance from point 2 to point 4 be represented by y, and the distance from point 3 to point 4 be represented by z. The triangle 1–3–4 of the space diagram is similar to the triangle *dac* of the force diagram; and the triangle 2–4–3 of the space diagram is similar to the triangle *dcb* of the force diagram. From the homologous sides of 1–3–4 and *dac*,

$$\frac{x}{dc} = \frac{z}{P}. \qquad (3)$$

From the homologous sides of 2–4–3 and *dcb*,

$$\frac{y}{dc} = \frac{z}{Q}. \qquad (4)$$

From Equations (3) and (4),

$$\frac{x}{y} = \frac{Q}{P}; \qquad (5)$$

$$Px = Qy. \qquad \text{Formula VI.}$$

The angles α and θ do not appear in any of the equations above.

It is evident, therefore, that the magnitude, direction, and position of the equilibrant (and resultant) are independent of the form of the jointed frame. These are the same whether the rigid body is a jointed frame, as in Fig. 75, or a single beam, as in Fig. 74. The triangular frame is merely a convenient means of finding the equilibrant.

This same result may be obtained by resolving the forces **P** and **Q**, with one component of **P** equal and opposite to one component of **Q**, and with these two components along the same line. These two components balance each other. The remaining components intersect. Their resultant at the point of intersection may be found by the methods for concurrent forces.

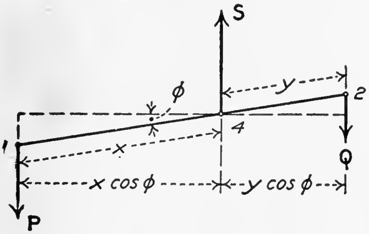


FIG. 77.

In Fig. 77, a line is drawn through the point 4 at right angles to the direction of **P** and **Q**. This line makes an angle ϕ with the direction of the line 1–2. Multiplying both sides of Formula VI by $\cos \phi$,

$$Px \cos \phi = Qy \cos \phi. \tag{6}$$

Since $x \cos \phi$ is the perpendicular distance from the point 4 to the line of the force **P**, the term $P x \cos \phi$ is the moment of the force **P** about the point 4 (or about any point in the line of the equilibrant). $Qy \cos \phi$ is, likewise, the moment of the force **Q** about point 4. Equation (6) may be interpreted as follows: *The moment of one of the parallel forces about any point in the line of the resultant is equal and opposite to the moment of the other force about that point.*

The moment of the resultant of two parallel forces about any point in their plane is equal to the sum of the moments of the two forces about that point. This may be proved from Fig. 78.

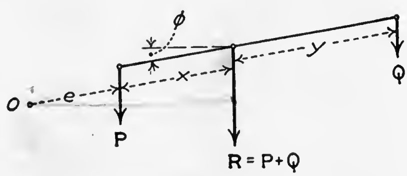


FIG. 78.

Take moments about the point O at a distance e from the line of the force **P** measured in the direction of x and y. The moment of the resultant about this point is

$$M_r = R(e + x) \cos \phi = (Pe + Px + Qe + Qx) \cos \phi. \tag{7}$$

The sum of the moments of **P** and **Q** about that point is given by the equation,

$$M = (Pe + Qe + Qx + Qy) \cos \phi. \tag{8}$$

Subtracting Equation (8) from Equation (7),

$$M_r - M = (Px - Qy) \cos \phi \tag{9}$$

Since $Px = Qy$, $M_r - M = 0$, and $M_r = M$. This equation proves the proposition.

If there are three forces in the same direction, two of these forces may be replaced by their resultant. This resultant is then combined with the third force to get the resultant of all three. This process may be continued indefinitely. *The resultant of any number of parallel forces in the same direction is equal to the sum of the forces, and the moment of the resultant about any point is equal to the sum of the moments of the separate forces about that point.* When all the distances are measured along parallel lines, the term $\cos \phi$ may be omitted from the moment equation.

Example I

Two forces of 12 pounds and 18 pounds are parallel, in the same direction, and 5 feet apart. Find the magnitude and position of their resultant.

Resultant $= 12 + 18 = 30$ lb.

Taking moments about a point in the line of the force of 12 pounds, Fig. 79,

$$12 \times 0 = 0$$
$$18 \times 5 = 90$$
$$30x = 90$$
$$x = 3 \text{ ft.}$$

The resultant is a force of 30 pounds at a distance of 3 feet from the force of 12 pounds.

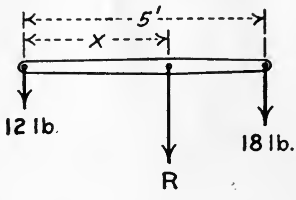


FIG. 79.

Example II

Given the following forces, all of which are vertically downward: 8 pounds at 2 feet, 12 pounds at 5 feet, 16 pounds at 10 feet, 9 pounds at 12 feet, and 5 pounds at 14 feet. Find their resultant.

The solution is best arranged in this form:

Force, lb.	Moment arm, ft.	Moment, ft.-lb.
8	2	16
12	5	60
16	10	160
9	12	108
5	14	70
50		414

$$50x = 414$$
$$x = 8.28 \text{ ft.}$$

The resultant is 50 pounds at a distance of 8.28 feet from the line of the 8-pound force.

Problems

1. Check Example II by moments about a point 2 feet from the 8-pound force.

2. A weightless, horizontal bar is 12 feet long and carries 40 pounds at the left end, 55 pounds 3 feet from the left end, 80 pounds 6 feet from the left end, and 25 pounds at the right end. Find the magnitude of the resultant and find its position by moments about the left end. Check by moments about some other point.

3. A beam 20 feet long weighs 40 pounds. Its center of mass is 9 feet from the left end. The beam is placed in a horizontal position and loaded with 60 pounds on the left end, 50 pounds 6 feet from the left end, 80 pounds 4 feet from the right end, and 30 pounds on the right end. The beam rests on a single support. Where must this support be placed?

4. A weightless bar 5 feet long forms a part of a jointed frame as in Fig. 76. The bar is horizontal and carries a load of 60 pounds on the left end and a load of 40 pounds on the right end. Solve graphically for the magnitude and position of the equilibrant. Make the angle $\alpha = 30°$ and use the scale of 1 inch = 1 foot on the space diagram, and 1 inch = 20 pounds on the force diagram.

5. Solve Problem 4 with $\alpha = 45°$.

6. Check Problem 4 by moments.

7. The ends of a bar 5 feet in length are connected to the ends of a chain which is 7 feet in length. A load of 50 pounds is hung on one end of the bar and a load of 30 pounds is hung on the other end. The chain is suspended from a hook. What is the length of chain between the hook and the 30 pound load if the bar hangs horizontal?

56. Resultant of Parallel Forces in Opposite Directions. In Fig. 76 or Fig. 77, the force **Q** may be regarded as the equilibrant of the forces **P** and **S**. Numerically

$$Q = S - P.$$

The resultant of the forces **P** and **S** in opposite directions is the force $\mathbf{R}_1$ of Fig. 80. This resultant is equal and opposite the force **Q**. From the equations of the preceding article, it is evident that the sum of the moments of **P, Q,** and **S** about any point is zero. Since $\mathbf{R}_1$ is equal and opposite to **Q,** the moment of $\mathbf{R}_1$ is equal to the sum of the moments of **S** and **P** taken with their proper signs. *The moment of the resultant of two parallel forces is equal to the sum of the moments of the forces, whether the forces are in the same direction or in opposite directions.*

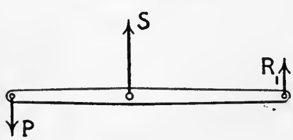


FIG. 80.

Example I

Find the magnitude and position of the resultant of a downward force of 8 pounds and an upward force of 18 pounds, the horizontal distance between the lines of the two forces being 5 feet.

Taking moments about a point in the line of the 8-pound force, and giving the negative sign to the downward force,

$$\begin{aligned} -8 \times 0 &= 0 \\ 18 \times 5 &= 90 \\ \hline 10x &= 90 \\ x &= 9 \text{ ft.} \end{aligned}$$

The resultant is a force of 10 pounds at a distance of 9 feet from the force of 8 pounds and 4 feet from the force of 18 pounds. The resultant is upward. A *downward* force of 10 pounds along the same line would balance the upward force of 18 pounds and the downward force of 8 pounds.

The resultant of any number of parallel forces is the algebraic sum of the forces; the moment of the resultant about any point is the algebraic sum of the moments of the several forces about that point.

Example II

Find the resultant of 10 pounds down at 2 feet, 17 pounds down at 5 feet, 21 pounds up at 10 feet, 16 pounds down at 12 feet, and 12 pounds down at 15 feet. Solve by moments about the 5-foot position. Call downward direction positive, since most of the forces are downward. With distances toward the right from the origin regarded as positive, clockwise moment is now positive.

Force, lb.	Arm, ft.	Moment, ft.-lb.
10	−3	−30
17	0	0
−21	5	−105
16	7	112
12	10	120

$$34x = 97$$
$$x = 2.85 \text{ ft.}$$

Problems

1. Solve Example II by moments about the 0-foot position.

2. Given the following vertical forces in the same vertical plane: 20 pounds down at 0 feet, 16 pounds up at 4 feet, 12 pounds up at 8 feet, and 10 pounds down at 10 feet. Find the magnitude and position of the resultant and check.

3. A horizontal beam 20 feet in length weighs 240 pounds. Its center of mass is at the middle of its length. It carries a load of 160 pounds 5 feet from the left end, and a load of 120 pounds at the right end. The left end rests on a platform scale which reads 170 pounds. There is a second support near the right end. Where is it located, and what load does it carry?

4. By the graphical method, find the resultant of a downward force of 6 pounds and an upward force of 10 pounds at a distance of 4 inches apart.

57. Equilibrium of Parallel Co-planar Forces.—Since the moment of the resultant of several parallel forces is equal to the sum of the moments of the separate forces, the moment of the equilibrant is equal and opposite to this sum. Consequently, if the body is in equilibrium, the algebraic sum of the moments of all the forces is zero, and the sum of the forces is zero. These relations may be expressed by the equations,

$$\Sigma P = 0;$$
$$\Sigma M = 0.$$

There are two unknowns. The problem may be solved by one moment equation and one resolution parallel to the direction of the forces. The resolution equation may be replaced by a second moment equation.

Example

A horizontal beam is 20 feet long. It weighs 240 pounds and its center of mass is at the middle of its length. The beam is supported at the left end and at 4 feet from the right end. It carries a load of 120 pounds 6 feet from the left end, and a load of 160 pounds at the right end. Find the reaction at each support.

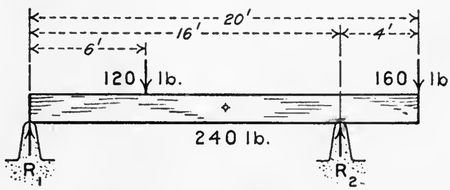


FIG. 81.

Take moments about the left support, in order to eliminate the reaction R_1, Fig. 81. Call downward force positive and distance toward the right positive.

Force, lb.	Arm, ft.	Moment, ft.-lb.
240	10	2400
120	6	720
160	20	3200
520		6320

$$16R_2 = 6320,$$
$$R_2 = 395 \text{ lb.}$$

Take moments about the right support and call distance toward the left positive.

Force, lb.	Arm, ft.	Moment, ft.-lb.
240	6	1440
120	10	1200
160	−4	−640
520		2000

$$16R_1 = 2000,$$
$$R_1 = 125 \text{ lb.}$$

Check by vertical resolutions; $395 + 125 = 520$ lb.

Problems

1. A horizontal beam 24 feet long, with its center of mass at the middle, weighs 180 pounds. It is supported 4 feet from the left end and 2 feet from the right end. It carries 144 pounds on the left end, 160 pounds 13 feet from the left end, 162 pounds 6 feet from the right end, and 90 pounds on the right end. Find the reactions at the supports and check.

2. The beam of Problem 1 is supported 2 feet from the left end. A second support near the right end exerts an upward force of 400 pounds. Where is this second support located, and what is the reaction of the left support?

3. A beam 20 feet long, with its center of mass at the middle of its length, weighs 160 pounds. It is supported 5 feet from the left end and is held down by a force one foot from the left end. The beam carries a load of 120 pounds on the right end. Find the reactions and check.

58. Condition of Stable Equilibrium.—When a body is in equilibrium under the action of non-concurrent forces, insofar as the equilibrium in that position is concerned, any one of the forces may be regarded as applied to the body at any point along its line of action. If, however, the equilibrium be disturbed, either by a slight displacement of the body, or by a small change in the magnitude of one of the forces, then the position of application of the forces is the element which determines whether the body will take a new position with a small displacement, or will continue to move through a greater distance. The position of application of the forces determines whether the equilibrium is stable, unstable, or neutral.

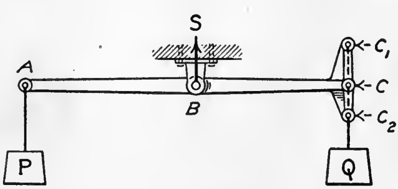


FIG. 82.

Figure 82 shows a rigid body which is supported at B and loaded at A. A second load Q acts along the line through C_2, C and C_1. If the body is in equilibrium in this position, it is in equilibrium whether the load is applied at C, at C_1, or at C_2. The points A, B, and C lie in a straight line, as shown in Fig. 83, I. If the beam is turned through an angle θ from the horizontal position, the moment arm of the force **P** becomes $x \cos \theta$ and the moment arm of **Q** becomes $y \cos \theta$. In the original position of equilibrium

$$Px = Qy;$$

consequently

$$Px \cos \theta = Q\, y \cos \theta.$$

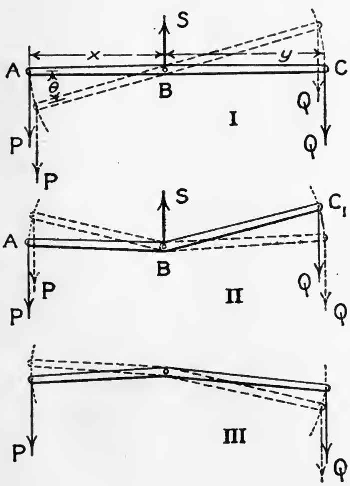


FIG. 83.

There is the same relative change in the moments of the two forces; therefore, the beam is in equilibrium in the new position.

When a body is in equilibrium under the action of three parallel forces, if the points of application of the three forces lie in a straight line, the equilibrium is neutral.

Figure 83, II, represents the case in which the load is applied at a point C_1 placed above the line through A and B. The broken lines illustrate the condition when the beam is rotated in a clockwise direction about B. The moment arm of **Q** becomes longer and the moment arm of **P** becomes shorter. After the arm BC has passed the horizontal position, the moment arm of **Q** diminishes, but less rapidly than the effective arm of the force **P**. If the forces **P** and **Q** in the original position produced equal and opposite moments about B, the moment of **Q** in any one of these new positions is greater than the moment of **P**, and the beam continues to rotate about B through approximately 180 degrees to the position of stable equilibrium. The equilibrium of Fig. 83, II, is *unstable.* If the beam had been rotated in the opposite direction from the position of equilibrium, the same condition would obtain and it would continue to rotate in that direction.

Figure 83, III, represents the case in which the load is applied at C_2 placed below the line through A and B. When the beam is rotated slightly from this position in a clockwise direction, the moment arm of **P** becomes a little longer and that of **Q** becomes a little shorter. The beam will return to its original position after displacement. The equilibrium is *stable.*

Example

Figure 84 represents a rectangular board in a vertical plane supported by a smooth hinge at the middle. A load of 10 pounds is suspended from A_2 and a load of **Q** pounds is suspended from C_2. The board rotates through an angle of 10 degrees in a clockwise direction. Find the load **Q**.

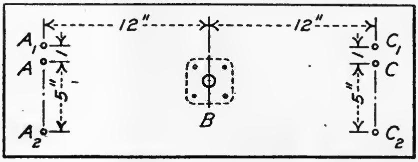


FIG. 84.

It is assumed that the center of mass is at the center of the board, so that its weight exerts no moment about the support in any position. Taking moments about the support,

$$10 \times 13 \cos 12° \, 37' = Q \times 13 \cos 32° \, 37'; \text{ or}$$
$$10\,(12 \cos 10° + 5 \sin 10°) = Q\,(12 \cos 10° - 5 \sin 10°)$$

Problems

1. In Fig. 84, the load at A_2 is 10 pounds and the load at C_2 is 12 pounds. Find the position of equilibrium. *Ans.* The rotation is 12° 36′ clockwise.

2. Solve Problem 1 if the loads are 20 pounds and 22 pounds, respectively.

3. In Fig. 84, the board weighs 4 pounds and its center of mass is 2 inches below the point of support. The load **P** is attached at *A* and the load **Q** at *C*. The points *A*, *B*, and *C* are in a horizontal straight line. Find the position of equilibrium if **P** is 10 pounds and **Q** is 11 pounds and there is no friction at *B*. *Ans.* Rotation $= \tan^{-1} \frac{3}{2} = 56° \, 19'$.

4. Solve Problem 3 for loads of 5 pounds and 6 pounds, respectively.

5. In Fig. 84, the loads are suspended from $\mathbf{A}_1$ and $\mathbf{C}_1$, which are 1 inch above *A* and *C*, respectively. Find the position of equilibrium when **P** is 1 pound and **Q** is 2 pounds. *Ans.* Rotation $= \tan^{-1} 2.4 = 67° \, 23'$.

6. Solve Problem 5 for loads of 4 pounds and 5 pounds.

7. In Fig. 84, a load of 10 pounds is applied at A_2 and a load of 11 pounds at C_2. The board weighs 4 pounds and its center of mass is 2 inches below *B*. Find the position of equilibrium. *Ans.* Rotation = 6° 04′.

8. Solve Problem 7 if the loads are 1 pound and 2 pounds.

59. Resultant of Non-parallel Forces Graphically.—Figure 85 shows three forces, which are supposed to be applied to a rigid body. These forces are: 20 pounds at 30 degrees with the horizontal, applied at the point (5, 0); 15 pounds at 75 degrees with the horizontal, applied at the point (3, 2); and 12 pounds at 115 degrees with the horizontal toward the right, applied at the point $(-2, 3)$. It is desired to find the magnitude and direction of the resultant, and the position of the line along which it acts.

Figure 85, II, is the force diagram. The forces of 20 pounds and 15 pounds are laid off and their resultant is found. This resultant is $\mathbf{R}_1$. Then on the space diagram the lines of the 20-pound force and the 15-pound force are extended till they intersect at B. Through B a line is drawn parallel to the direction of $\mathbf{R}_1$ of the

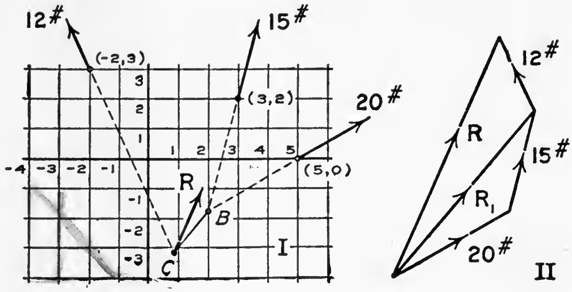


FIG. 85.

force diagram. The resultant of the forces of 20 pounds and 15 pounds may be regarded as acting along this line. A vector representing the force of 12 pounds is now added to $\mathbf{R}_1$ of the force diagram. The vector sum is $\mathbf{R}$, which represents the direction and magnitude of the resultant of all three forces. On the space diagram, the line of the 12-pound force is extended till it intersects the line of $\mathbf{R}_1$ at C. Through C a line is drawn parallel to $\mathbf{R}$ of the force polygon. This line gives the position of the resultant of all three forces.

Problem

Given the following forces: 20 pounds horizontal toward the right through the point (2, 4); 25 pounds at an angle of 35 degrees to the right of the vertical through the point (3, 3); 15 pounds vertical through the point (1, 3); and 16 pounds at an angle of 45 degrees to the left of the vertical through the point $(-2, 2)$. Find the direction and magnitude of the resultant by means of the force polygon, and find its position on the space diagram. Use 1 inch = 5 pounds on the force diagram, and 1 inch = 2 units of length on the space diagram. Measure the perpendicular distance from the point (0, 0) to the line of the resultant.

60. Calculation of the Resultant of Non-parallel Forces.—The force polygon of Fig. 85, II, is exactly the same as that for a set of concurrent forces. It is evident, therefore, that the magnitude and direction of the resultant of non-concurrent forces may be calculated by the method of Art. 42. Each force is resolved into two components along a pair of axes at right angles to each other. The sum of the components along one axis is taken as one side of right-angled triangle. The sum of the components along the other axis is taken as the other side of the same right-angled triangle. The resultant is represented by the hypotenuse of his triangle.

The position of the line of action of this resultant is calculated by moments. In Fig. 85, the force of 20 pounds and the force of 15 pounds may be regarded as concurrent at B. According to Art. 48, the moment of $\mathbf{R}_1$ about any point is equal to the sum of the moments of the 20-pound force and the 15-pound force about that point. The resultant $\mathbf{R}_1$ may be regarded as concurrent with the force of 12 pounds at C. The sum of the moments of $\mathbf{R}_1$ and the 12-pound force about any point is equal to the moment of the final resultant about that point. *The moment about any point of the resultant of a set of non-concurrent, coplanar forces is equal to the sum of the moments of the several forces about that point.*

Example

A rectangular board, Fig. 86, is 4 feet wide and 3 feet high. A force of 20 pounds, at an angle of 15 degrees with the horizontal toward the right,

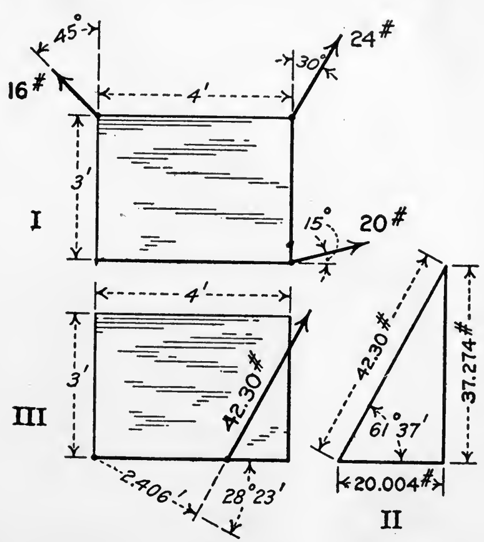


FIG. 86.

is applied at the lower right corner. A force of 24 pounds, at an angle of 30 degrees to the right of the vertical, is applied at the upper right corner. A force of 16 pounds at an angle of 45 degrees to the left of the vertical is applied at the upper left corner. Find the magnitude and direction of the resultant, and its distance from the lower left corner.

Resolving horizontally and vertically,

Force	H component	V component
20	19.318	5.176
24	12.000	20.784
16	−11.314	11.314
	20.004	37.274

The resultant is the hypotenuse of the right-angled triangle of which the base is 20.004 units and the altitude is 37.274 units. The resultant makes an angle of 61° 47′ with the horizontal toward the right. Its magnitude is 42.30 pounds.

To find the location of the line of the resultant force on the space diagram, moments are taken about the lower left corner of the board. Since the horizontal and vertical components of the several forces have already been computed, it is convenient to use these components in calculating the moments.

The horizontal component of the 20-pound force and the vertical component of the 16-pound force pass through the origin of moments so that the moment of each of these components is zero.

V component of 20 lb.	$5.176 \times 4 =$	20.704
V component of 24 lb.	$20.784 \times 4 =$	83.136
H component of 24 lb.	$12.000 \times 3 =$	− 36.000
H component of 16 lb.	$11.314 \times 3 =$	33.942

$$M = 101.782 \text{ ft.-lb.}$$

$$\frac{101.782}{42.30} = 2.406 \text{ ft.}$$

Figure 86, III, shows the location of the resultant. Since the resultant makes an angle of 61° 47′ with the horizontal, a line perpendicular to it makes an angle of 28° 13′ with the horizontal. To locate the resultant on the space diagram, a line is first drawn through the lower left corner at an angle of 28° 13′ below the horizontal toward the right. A length of 2.406 feet is measured off on this line from the origin of moments to the point B. The moment is counter-clockwise and the resultant force is upward; hence the moment arm must be measured toward the right from the corner of the board. Through the point B, a line is drawn perpendicular to OB. The resultant force is applied along this perpendicular line. If a smooth pin were passed through the board at any point along this line, this pin would hold the board in equilibrium, and the reaction at the point of contact would be 42.30 pounds. The equilibrium would be stable, unstable or neutral, depending upon the position of the pin in the line of action of the resultant force.

It is best to determine the sign of the moments by observing on the space diagram whether the rotation is clockwise or counter-clockwise, rather than by using the signs of the forces and effective arms. A *horizontal* force toward the right applied to a vertical arm upward gives a clockwise moment. A vertical force upward applied to a horizontal arm toward the right gives a counter-clockwise moment. If counter-clockwise moment be taken as positive,

$$M = Vx - Hy,$$

in which V is the component of the force parallel to the Y axis, H is the component of the force parallel to the X axis, x is the abscissa, and y is the ordinate of some point in the line of action of the force. It is not necessary that the X axis be horizontal and the Y axis vertical. The X axis may have any direction and the Y axis be perpendicular to the X axis.

The student who prefers to make use of the signs of the forces and distances may employ the equation above, but he should also check his results and visualize his problem by observing the direction of rotation on the space diagram.

Instead of finding the perpendicular distance from the origin of moments to the line of the resultant, the points of intersection of the resultant force with the axes of coördinates might have been calculated. Since the origin of moments lies in the X axis, the moment of the horizontal component is zero at the point where the line of the resultant cuts that axis.

$$101.782 = Vx,$$

in which x is the abscissa of the point of intersection with the X axis.

$$x = \frac{101.782}{37.274} = 2.731 \text{ ft.}$$

Regarding the force as applied at the Y intercept,

$$y = \frac{101.782}{20.004} = 5.088 \text{ ft.}$$

The moment is counter-clockwise. A vertical force upward gives a counter-clockwise moment when the point of application is to the right of the origin of moments. A horizontal force toward the right gives a counter-clockwise moment when the point of application is below the origin.

Problems

1. Solve the Example of Fig. 85 by resolutions and moments.

2. Solve the Problem of Art. 59 and compare with your graphical solution.

3. Find the direction, magnitude, and the line of application of the resultant of the following forces: 20 pounds, at 65 degrees to the right of the vertical, through the point (2, 3); 16 pounds, at 20 degrees to the right of the vertical, through the point (−2, 2); 12 pounds, at 40 degrees to the left of the vertical, through the point (1, 5); and 18 pounds, at 110 degrees to the left of the vertical, through the point (−3, 2).

The solution may be arranged in tabular form as was done with Problem 4 of Art. 17.

Force, lb.	Angle, deg.	x	y	$\cos \alpha$	$\sin \alpha$	H	V	Vx (counter-clockwise +)	$-Hy$
20	25	2	3	0.9063	0.4226	18.126	8.452	16.904	−54.378
16	70	−2	2	0.3420	0.9397	5.472	15.035	−30.070	−10.944
12	130	1	5	−0.6428	0.7660	− 7.714	9.192	9.192	38.570
18	200	−3	2	−0.9397	−0.3420	−16.915	−6.156	18.468	33.830
						− 1.031	26.523	14.494	7.078
									14.494
						Total moment			21.572

The resultant force is 26.55 pounds at an angle of 2° 14′ to the left of the vertical. The total moment is 21.572 units counter-clockwise. The resultant force passes through a point at a distance of 0.812 units from the origin, measured along a line which makes an angle of 2° 14′ above the horizontal toward the right.

4. In Problem 3, find the point where the resultant cuts the X axis, by dividing the total moment by the vertical component of the resultant force.

Ans. $x = 0.813$.

5. In Problem 3, find the point where the resultant cuts the Y axis.

6. In Problem 3, compute the moments about the point (2, 2), instead of the origin, and find the points of intersection of the resultant force with the lines $x = 2$ and $y = 2$.

61. Equilibrium of Non-concurrent Forces.—A set or non-concurrent forces acting on a rigid body may be divided into two groups. In order that equilibrium may exist, two conditions must be satisfied. These are:

1. The resultant of the forces of one group must be equal and opposite to the resultant of all the other forces.

2. The resultant of the forces of one group must lie along the same line as the resultant of all the other forces.

The first condition is identical with the condition of equilibrium of concurrent forces. Stated graphically, this means that the force polygon must close. Stated algebraically, this means that the sum of the components of all the forces along any direction is zero. Literally it is written,

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + \text{etc.} = 0, \qquad (1)$$

$$\Sigma P \cos \alpha = 0. \qquad (2)$$

In the case of coplanar forces, two independent equations of this kind may be written.

The resultant of one group of forces may be equal and opposite to the resultant of all the other forces, but may not lie in the same line. In the latter case, the moment of one group of forces will not balance the moment of the other forces, and the forces will rotate the body. The second condition is satisfied when the sum of the moments of all the forces which act on the rigid body is zero. This condition is expressed algebraically by the equation

$$\Sigma M = 0. \tag{3}$$

A problem of the equilibrium of non-concurrent, coplanar forces may be solved by writing *two resolution equations and one moment equation.* Since a moment equation may replace a resolution equation, any one of the following combinations may be used:

1. One moment equation and two resolution equations.
2. Two moment equations and one resolution equation.
3. Three moment equations.

As in the case of concurrent forces, resolution perpendicular to the line of action of a force eliminates that force; moment about a point in the line of action of a force eliminates that force. Moment about the point of intersection of two forces eliminates both of them. When two forces of unknown magnitude intersect, it is advisable to begin the solution by writing the moment equation about their point of intersection as the origin. Frequently, both the direction and magnitude of one force are unknown. If, however, some point on the line of action of this force is known, it is advisable to take moments about this point.

Example

A bar AB, Fig. 87, is 20 feet long, weighs 60 pounds, and has its center of mass 8 feet from A. The bar is hinged at A and supported by a cord at B. The bar makes an angle of 15 degrees above the horizontal toward the right, and the cord makes an angle of 35 degrees to the left of the vertical. Find the tension in the cord and the direction and magnitude of the hinge reaction at A.

The *free body* is the entire bar AB. The forces which act on the free body are its weight, the unknown tension in the cord at B, and the reaction of the hinge at A. The hinge reaction is unknown, as to both direction and magnitude.

Begin by writing a moment equation about the hinge A, since this eliminates two unknowns.

$$60 \times 8 \times \cos 15° = P \times 20 \cos 20° \tag{4}$$
$$24 \times 0.9659 = 0.9397P$$
$$P = 24.67 \text{ lb.}$$

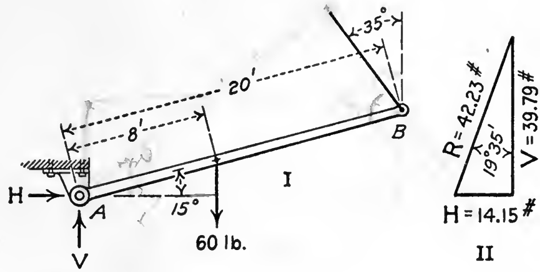


FIG. 87.

To find the direction and magnitude of the reaction at the hinge, it is convenient to regard it as made up of a horizontal component **H** and a vertical component **V**. Resolving horizontally,

$$H = P \sin 35° = 14.15 \text{ lb.} \tag{5}$$

Resolving vertically,

$$V = 60 - P \cos 35° = 60 - 20.21 = 39.79 \text{ lb.} \tag{6}$$

From the force triangle, Fig. 87, II, the resultant reaction of the hinge is found to be 42.23 pounds at an angle of 19° 35′ to the right of the vertical.

Check all results by moments about B,

$$60 \times 12 \cos 15° = 42.23 \times 20 \cos 34° 35' \tag{7}$$

Problems

1. Complete the check of the Example above by substituting in Equation (7).

2. Solve the Example of Fig. 87 if the cord makes an angle of 35 degrees to the right of the vertical. Check the result.

3. Consider the hinge reaction in Fig. 87 as made up of a component parallel to AB and a component perpendicular to AB. Find the component perpendicular to AB by moments about B. Find the component parallel to AB by a resolution parallel to the bar. Calculate the resultant reaction and check.

4. A vertical door, Fig. 88, is 12 feet wide, 10 feet high, and weighs 360 pounds. The hinges are one foot from the top and bottom, respectively, and are so placed that all the vertical load comes on the lower hinge. Find the direction and magnitude of each hinge reaction.

5. A derrick mast is 30 feet long. The boom is 50 feet long, is elevated

20 degrees above the horizontal, and carries a load of 1200 pounds. A guy rope attached to the top of the mast makes an angle of 35 degrees with the horizontal. Considering the mast, boom, and the ropes which connect them as a single rigid body, find the tension in the guy rope, and the direction and magnitude of the reaction at the base due to the load of 1200 pounds.

6. A derrick mast is 15 feet long. The boom is 20 feet long and is elevated to a position in which the free end is 16 feet from the top of the mast. A guy rope in the same plane as the boom and mast is attached to a point at the same level as the bottom of the mast, at a distance of 18 feet from the bottom. Find the tension in the guy rope and the horizontal and vertical components of the reaction at the bottom of the mast due to a load of 1600 pounds on the end of the boom. Construct the space diagram to the scale of 1 inch = 5 feet, and solve by measuring moment arms on the diagram.

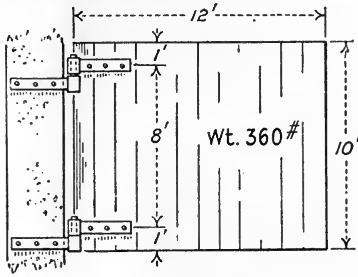


FIG. 88.

7. In Problem 6, calculate the angles and the moment arms trigonometrically, and solve by one moment and two resolution equations.

8. Solve Problem 6 for all forces by Bow's method, drawing the lines in the force diagram parallel to the corresponding members of the space diagram.

9. Solve Problem 5 for all forces joint at a time by resolutions.

10. A bar *AB*, Fig. 89,[1] is 7 feet long, weighs 28 pounds, and has its center of mass 3 feet from *A*. The end *A* is provided with a cylindrical roller,

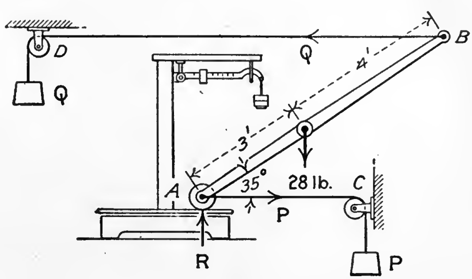


FIG. 89.

which allows it to move on a surface with little friction. The bar is placed with the end *A* on a horizontal platform. It is supported by a cord *BD*

[1] The apparatus as shown in Fig. 89 is decidedly unstable. One of the masses *P* or *Q* should be free to move only a short distance. The mass might be placed on a second platform scale and the difference of weight taken. To avoid lateral instability, the pulley at *D* should be to the right of the vertical line through *A*.

attached at B and held at A and by a second cord AC. Find the tension in each cord and the reaction of the platform when the bar makes an angle of 35 degrees with the horizontal and both cords are horizontal.

Ans. $P = Q = 17.14$ lb.; $R = 28$ lb.

11. Solve Problem 10 if AC is horizontal, BD makes an angle of 15 degrees with the horizontal, and the point D is higher than B.

Ans. $Q = 12.83$ lb.; $P = 12.39$ lb.; $R = 24.68$ lb.

12. Solve Problem 10 if AC is horizontal, BD makes an angle of 15 degrees with the horizontal, and the point D is lower than B.

13. Solve Problem 10 if BD is perpendicular to AB, and also, if BD is vertical.

14. In Fig. 89, the cords AC and BD are horizontal and the force **P** is 10 pounds. Find the angle which the bar makes with the horizontal.

15. A bar AB is 10 feet long, weighs 30 pounds, and has its center of mass 4 feet from A. The end A rests on a horizontal floor and the end B rests against a vertical wall. Both ends are so constructed as to have practically no friction. The bar is held by a cord attached 1 foot from A. Find the tension in the cord and the reactions at the floor and at the wall when the bar makes an angle of 30 degrees with the vertical and the cord is horizontal. In what respect does this differ from Problem 10?

16. Solve Problem 15 if the wall makes an angle of 15 degrees with the vertical, away from the bar, as shown in Fig. 90.

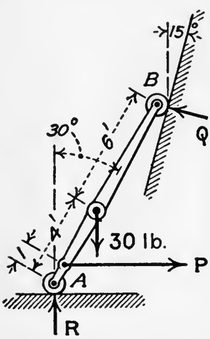


FIG. 90.

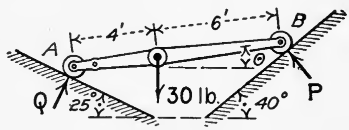


FIG. 91.

In order to eliminate two unknowns, it is advisable to take moments about the point of intersection of the vertical line through A with the line of the horizontal cord. It is best to use the horizontal and vertical components of Q in this moment equation. A part of the solution is

$$30 \times 4 \sin 30° = Q(9 \cos 30° \cos 15° + 10 \sin 30° \sin 15°),$$
$$Q = 6.800 \text{ lb},$$
$$P = Q \cos 15° = 6.569 \text{ lb}.$$

The student will verify these results and check by moments about A. He should find **R** by vertical resolution and check by moments about B.

17. The bar of Problem 15 is placed upon two inclined planes, Fig. 91. The end A rests on a plane which makes an angle of 65 degrees to the left of the vertical, and the end B rests on a plane which makes an angle of 50 degrees to the right of the vertical. Find the reaction of each plane and the angle which the bar makes with the horizontal.

Compute the reactions by resolving parallel to the inclined planes. Then find the angle by moments about A. It is best to use the components of the force **P** in the moment equation.

$$30 \times 4 \cos \theta = P \sin 40° \times 10 \sin \theta + P \cos 40° \times 10 \cos \theta,$$
$$12 = P (\sin 40° \tan \theta + \cos 40°).$$

Check by moments about B. *Ans.* $\theta = 8° 07'$.

62. Condition for Independent Equations.—*If two moment equations of equilibrium are written for a problem of non-concurrent, coplanar forces, and these equations are combined into a single equation, the resulting equation is equivalent to a resolution perpendicular to the line joining the two origins of moment.*

In Fig. 92, A and B are taken as the two origins of moment. The point B is the origin of coordinates and the X axis is passed through the point A. The distance between A and B is equal to c. A force $\mathbf{P}_1$ at an angle α_1 with the X axis passes through the

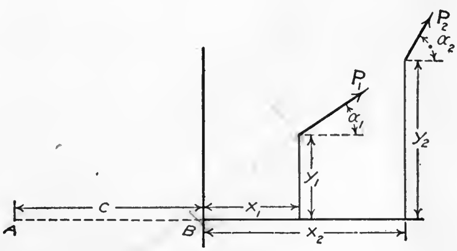


FIG. 92.

point (x_1, y_1) and a force P_2 at an angle α_2 with the X axis passes through the point (x_2, y_2), etc. Taking moments about B,

$$x_1P_1 \sin \alpha_1 - y_1P_1 \cos \alpha_1 + x_2P_2 \sin \alpha_2 - y_2P_2 \cos \alpha_2 + \text{etc.} = 0. \quad (1)$$

Taking moments about A,

$$(x_1 + c)P_1 \sin \alpha_1 - y_1P_1 \cos \alpha_1 + (x_2 + c)P_2 \sin \alpha_2 - y_2P_2 \cos \alpha_2 + \text{etc} = 0. \quad (2)$$

Subtracting Equation (1) from Equation (2),

$$c(P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + \text{etc.}) = 0. \quad (3)$$

The term in the parenthesis of Equation (3) is the sum of the components parallel to the Y axis; therefore, Equation (3) proves the proposition.

If two moment equations are written and one resolution equation, and if the resolution is taken along a direction which is perpendicular to the line joining the two origins of moment, the three equations will not be independent. If three moment equations are written, and if the three origins of moment lie on the same straight line, the three equations will not be independent.

For concurrent, coplanar forces in equilibrium there are *two* unknowns, and two independent equations may be written. These may be:

(1) Two resolutions,

(2) One moment and one resolution.

(3) Two moments.

For non-concurrent, coplanar forces there are three unknowns, and three independent equations may be written. These are the same as the equations for concurrent forces with the addition of one moment equation. If the forces are all parallel, there can be only one independent resolution equation, and only two unknown quantities.

When one resolution and *one* moment equation are written for concurrent forces, the resolution must not be taken perpendicular to the line which joins the origin of moments and the point of application of the forces. When one resolution and *two* moment equations are written for non-concurrent forces, the resolution must not be taken perpendicular to the line which joins the two origins of moment.

When *two* moment equations are written for concurrent forces, the two origins of moment and the point of application of the forces must not lie on the same straight line. When *three* moment equations are written for non-concurrent forces, the three origins of moment must not lie on the same straight line.

63. Direction Condition of Equilibrium.—When a body is in equilibrium under the action of three non-concurrent forces, in order that any one of these forces may lie in the line of the resultant of the other two forces, it is necessary that all three forces intersect at a single point. In Fig. 93, the bar AB is hinged at A and supported by a cord at B. The vertical line through the center of mass at the point C intersects the line of the cord at D. In order that the bar may be in equilibrium, the direction of the hinge reaction must be such that its line of action shall pass through D. This relation is sometimes called the

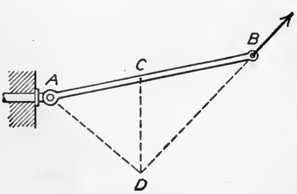


FIG. 93.

geometrical condition of equilibrium. In this book it will be called the *direction condition* of equilibrium.

Example I

A horizontal bar 5 feet long, weighing 30 pounds, with its center of mass 2 feet from the left end, is supported by cords at the ends. The cord at the left end makes an angle of 30° to the left of the vertical. Find the direction of the cord at the right end.

From the right-angled triangles of Fig. 94,

$$CD = 2 \tan 60° = 3 \tan \theta,$$

$$\tan \theta = \frac{2 \times 1.7321}{3} = 1.1547,$$

$$\theta = 49° 06'.$$

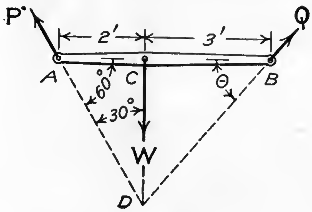


FIG. 94.

The cord at the right end makes an angle of 40° 54′ to the right of the vertical. The magnitude of the forces may be found by moments or resolutions.

The direction condition may take the place of one moment equation in the solution of a problem of non-concurrent forces. The other necessary equations may be resolutions.

This method of solution is especially valuable when only the direction is required. In practical work it is often desirable to know the direction of a force, in order to put in a support in the best position. The engineer should cultivate the habit of observing the direction of forces in actual structures.

Problems

1. A horizontal beam, 12 feet long, with its center of mass 5 feet from the left end, is supported by two posts. One post at the left end makes an angle of 25 degrees to the right of the vertical. What should be the direction of the post at the right end in order that the compression in each post shall be parallel to its length? Solve also if the second post is 2 feet from the right end.

2. Solve Problem 1 if one post is at the left end and the other is 3 feet from the left end.

3. A derrick boom is 20 feet long, makes an angle of 40 degrees with the horizontal, and carries a load of 400 pounds. The mast is 12 feet long. One guy rope, which is in the plane of the mast and boom, makes an angle of 30 degrees with the horizontal. Find the direction of the reaction at the bottom of the mast by means of the direction condition of equilibrium. With this direction known, draw the force triangle for the three external forces, which are the weight on the boom, the tension of the guy rope, and the reaction at the bottom of the mast. Also solve completely by Bow's method for all external and internal forces. Compare the two diagrams.

4. A bar 7 feet long, with its center of mass 3 feet from the left end, is supported by a rope at the left end which makes an angle of 30 degrees to the left of the vertical, and a rope at the right end which makes an angle of 35 degrees to the right of the vertical. What angle does the bar make with the horizontal? If the bar weighs 60 pounds, what is the tension in each rope?

From Fig. 95,

$$DF = DE + EF.$$
$$4 \cos \theta \tan 55^\circ = 3 \cos \theta \tan 60^\circ + 7 \sin \theta;$$
$$\tan \theta = \frac{4 \tan 55^\circ - 3 \tan 60^\circ}{7} = 0.0737;$$
$$\theta = 4^\circ\ 13'.$$

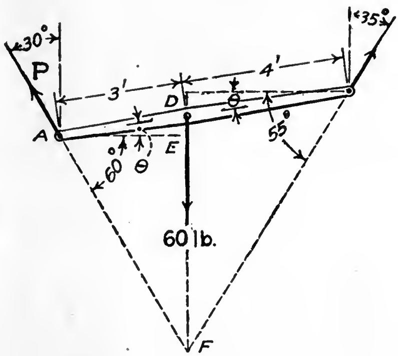


FIG. 95.

By a resolution perpendicular to the direction of the rope at the left end,

$$Q \cos 25^\circ = 60 \sin 30^\circ,$$
$$Q = 30 \sec 25^\circ = 30 \times 1.1034 = 33.10 \text{ lb.}$$

Check **Q** and θ by moments about the left end. Solve for **P** by a resolution equation and check.

5. Find the direction of the bar in Problem 17 of Art. 61 by means of the direction condition of equilibrium.

6. A bar 5 feet long, weighing 20 pounds, with its center of mass 2 feet from one end, is placed across the inside of a hollow cylinder which is 6 feet in diameter. The ends of the bar are frictionless. Find the position of equilibrium and the normal reactions at the ends.

Ans. The bar makes an angle of 16° 47′ with the horizontal.

When a body is in equilibrium under the action of four forces, the resultant of two of these forces must lie in the line of the resultant of the other two forces. In order that this may happen, the resultant of two of these forces must pass through the intersection of the lines of action of the others.

Example II

A ladder 20 feet long, weighing 40 pounds, with its center of mass 8 feet from the lower end, rests on a smooth horizontal floor and leans against a smooth vertical wall. It is held from slipping by a horizontal force of 12 pounds at the bottom. Find the position of equilibrium.

The resultant of the horizontal force and the vertical reaction at the bottom must pass through the point D, (Fig. 96), at which the vertical line through the center of mass intersects the horizontal line through the top of the ladder. By vertical resolution, the vertical reaction at the bottom is found to be 40 pounds. The resultant of 40 pounds and 12 pounds makes an angle with the vertical whose tangent is 0.3. From the figure,

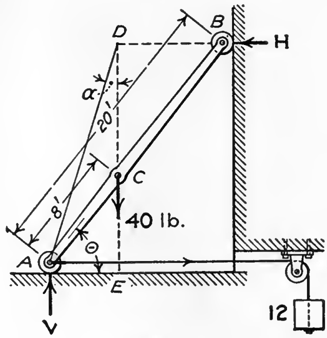


FIG. 96.

$$DE = 20 \sin \theta = 8 \cos \theta \cot \alpha;$$

$$\tan \theta = \frac{8}{0.3 \times 20} = 1.3333;$$

$$\theta = 53° \ 08'.$$

Problems

7. A bar 10 feet long, weighing 40 pounds, with its center of mass at the middle, rests on a smooth horizontal floor and leans against a smooth vertical wall. It is held by a horizontal pull of 10 pounds applied 2 feet from the lower end. Find the reactions at the floor and at the wall by resolutions, and find the angle which the bar makes with the vertical by means of the direction condition of equilibrium. *Ans.* 21° 48′ with the vertical.

8. A bar 12 feet long, with its center of mass 5 feet from the lower end, rests on a smooth horizontal floor and leans against a wall which makes an angle of 15 degrees with the vertical, away from the bar. The bar makes an angle of 35 degrees with the vertical. It is held by a horizontal force applied 2 feet from the bottom. Determine the three unknown forces graphically, using the direction condition of equilibrium to find the angles.

64. Trusses.—A truss is a jointed frame. Since a triangle is the only jointed frame which retains its form when loaded, a truss is made up of a connected series of triangular elements.

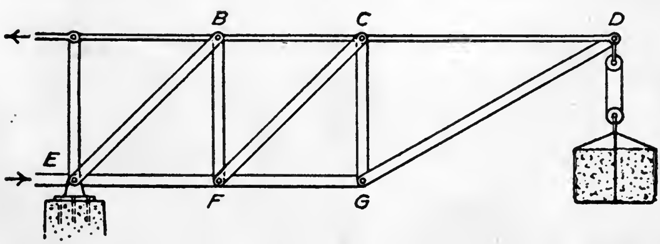


FIG. 97.

Figure 97 shows a truss which is entirely supported from one end. A truss supported at one end is called a *cantilever* truss. The distances *AB*, *BC*, and *CD*, are *panel* lengths. The top of the truss is called the *top chord*. The lower members, *EF* and *FG*, comprise the *bottom chord*. Either chord may be one, continuous, rigid body, but the calculations are made on the assumption that each panel is a separate body which is pin-connected to the remainder of the truss.

When loads are applied at the joints of a truss, the internal stresses are calculated as a series of problems of *concurrent* forces. Bow's method is used for the graphical solution. The truss of Fig. 97 may be calculated in this way, beginning at the right end, where there are only two unknowns.

In most trusses, no point can be found at which there are only two unknowns. It is then necessary to find first the external reactions as a problem of *non-concurrent* forces. After these reactions have been found, the forces at the separate joints are calculated as a series of connected members.

Example

The truss shown in the diagram of Fig. 98 is supported at the left end on rollers, which make the reaction vertical. It is hinged at the right end.

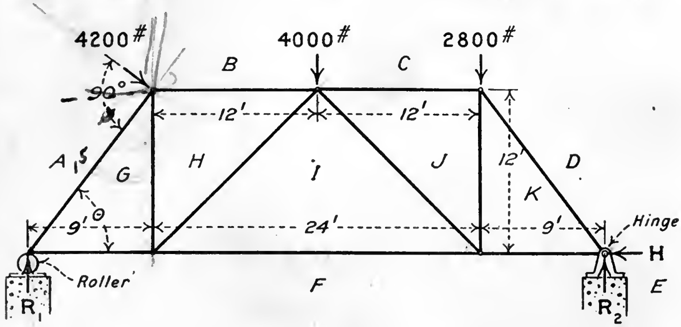


FIG. 98.

The reaction at the right end has a horizontal component **H** and a vertical component $\mathbf{R}_2$. Solve for the reactions and then solve for all the internal forces.

Moments about the left support eliminate $\mathbf{R}_1$ and **H**.

$$\begin{aligned} 4200 \times 15 &= 63000 \\ 4000 \times 21 &= 84000 \\ 2800 \times 33 &= 92400 \\ \hline 42R_2 &= 239400 \\ R_2 &= 5700 \text{ lb.} \end{aligned}$$

It is best to use the *components* of the load of 4200 pounds when moments are taken about the hinge. The end posts are 15 feet in length. Sin $\theta = \frac{12}{15} = 0.8000$; cos $\theta = 0.6000$. The force of 4200 pounds at right angles to the end post makes an angle θ with the horizontal. The horizontal component of 4200 pounds is $4200 \times 0.8 = 3360$ pounds. The vertical component is $4200 \times 0.6 = 2520$ pounds. Using these components instead of the 4200-pounds force, and taking moments about the right end of the truss,

$$\begin{array}{rcrl}
2800 \times 9 & = & 25200 & \\
4000 \times 21 & = & 84000 & \\
2520 \times 33 & = & 83160 & \\
\hline
 & & 192360 & \text{counter-clockwise,} \\
3360 \times 12 & = & 40320 & \text{clockwise,} \\
\hline
42R_1 & = & 152040 & \\
R_1 & = & 3620 & \text{lb.}
\end{array}$$

Resolve vertically for a check,

$$\begin{array}{rcr}
5700 & & 2520 \\
3620 & & 4000 \\
 & & 2800 \\
\hline
9320 & = & 9320
\end{array}$$

The horizontal component of the hinge reaction is obtained by a horizontal resolution,

$$\mathbf{H} = 3360 \text{ lb.}$$

This problem of non-concurrent, coplanar forces has now been solved by two moments and one resolution and partly checked by a second resolution, It might well be checked again by moments about the point of application of the 4200-pound load, or about the middle of the top chord.

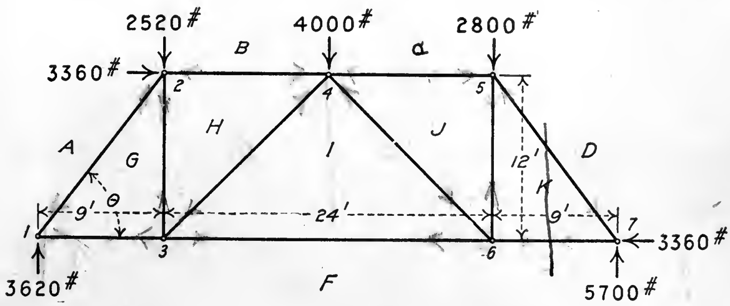


FIG. 99.

It is not best to use the components of the inclined force in both moment equations, for, if an error is made in the computation, the reactions will check and still be incorrect.

Figure 99 shows the reactions for the truss of Fig. 98. The force of 4200 pounds is replaced by its components. If a force and its components are written on the same diagram, one or the other should be enclosed in a parenthesis, or otherwise marked, on account of the danger of using both the force and its components in the same equation.

With the external forces known, the internal forces may now be computed. The algebraic method of solution, joint at a time, will be given first. This will be followed by the graphical solution. Figure 99 has been lettered by Bow's method. Each force will be represented by two letters. (It is most convenient in the algebraic method to use a single letter for each force. The two letters will be used here, however, in order to compare the results more readily with the graphical solution.)

Beginning with joint No. 1 as the free body and resolving vertically,

$$ag \sin \theta = 3620,$$
$$ag = \frac{3620}{0.8}, \qquad \textbf{ag = 4525 lb. compression.}$$

Resolving horizontally,

$$gf = ag \cos \theta = 4525 \times 0.6, \qquad \textbf{gf = 2715 lb. tension.}$$

The tension in GF might have been calculated by moments about joint No. 2,

$$gf \times 12 = 3620 \times 9.$$

At joint No. 2, AG pushes upward. Its horizontal component at the top is the same as at the bottom, or 2715 pounds. Its vertical component is 3620 pounds. Resolving horizontally,

$$bh = 3360 + 2715, \qquad \textbf{bh = 6075 lb. compression.}$$

Resolving vertically, and assuming that gh pulls downward,

$$ag \sin \theta = 2520 + gh,$$
$$gh = 3620 - 2520, \qquad \textbf{gh = 1100 lb. tension.}$$

Resolving vertically at joint No. 3,

$$hi \sin 45° = gh = 1100,$$
$$hi = 1100 \times 1.4142, \qquad \textbf{hi = 1556 lb. compression.}$$

Resolving horizontally, HI pushes toward the left and GF pulls toward the left, consequently IF must pull toward the right. The horizontal component of the force in HI is the same as the vertical component;

$$if = 2715 + 1100, \qquad \textbf{if = 3815 lb. tension.}$$

Resolving vertically at joint No. 4,

$$ij \sin 45° = 4000 - 1100 = 2900,$$
$$ij = 2900 \times 1.4142, \qquad \textbf{ij = 3701 lb. compression.}$$

Resolving horizontally, assuming that cj is compression,

$$cj = bh + hi \cos 45° - ij \cos 45°,$$
$$cj = 6075 + 1100 - 2900, \qquad \textbf{cj = 4275 lb. compression.}$$

At joint No. 5,

$$kd \cos \theta = cj = 4275,$$
$$kd = \frac{4275}{0.6}, \qquad \textbf{kd = 7125 lb. compression.}$$
$$jk + 2800 = kd \sin \theta = 7125 \times 0.8,$$
$$jk = 5700 - 2800, \qquad \textbf{jk = 2900 lb. tension.}$$

At joint No. 6,

$jk = ij \sin 45° = 2900$ lb.
$kf = if - ij \cos 45° = 3815 - 2900,$ **kf = 915 lb. tension.**

At joint No. 7,

$$kd \sin \theta = fe = R_2$$
$$R_2 = 7125 \times 0.8 = 5700 \text{ lb.}$$
$$kd \cos \theta - kf = de = H,$$
$$H = 4275 - 915 = 3360 \text{ lb.}$$

These last two results check the hinge reaction. The values of jk from joints 5 and 6 also check.

Figure 100 is the graphical solution for Fig. 98. The diagram begins with af, the vertical reaction at the left end. The last

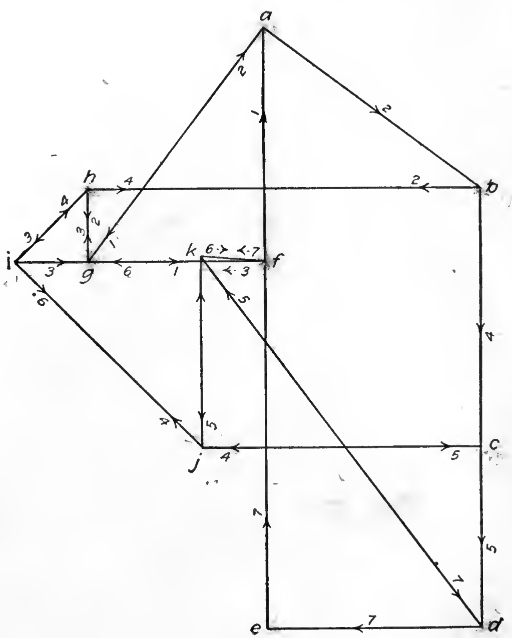


FIG. 100.

point is k. This point should fall on the horizontal line igf, in order that the line kf may be horizontal. On account of errors in the drawing, the point k is slightly above igf. The distance from k to the horizontal line is the *closing error.*

The external forces fa, ab, bc, cd, de, and ef form a polygon. A line representing an external force is recognized by the fact that it carries only one arrow. The external force polygon also must close.

Problems

1. Draw the space diagram, Fig. 98, to the scale of 1 inch = 8 feet, or 1 inch = 10 feet. With the external reactions known from the above example, draw the force diagram to the scale of 1 inch = 1000 pounds. Draw all forces parallel to the corresponding lines on the space diagram. Measure all lengths on the force diagram and put the values in pounds on the space diagram.

2. Solve the above Example by Bow's method, putting f at the top and a at the bottom of the first line af.

3. In Fig. 98, make the load in AB 2800 pounds, the load in BC 3200 pounds, and the load in CD 3500 pounds. Find the reactions and check. Solve for all internal forces, joint at a time. Solve by Bow's method to the same scale as Problem 1.

It is a good plan to carry Bow's method along with the algebraic solution for the internal forces. Calculate the forces at a given joint, and then draw the part of the force diagram for that joint.

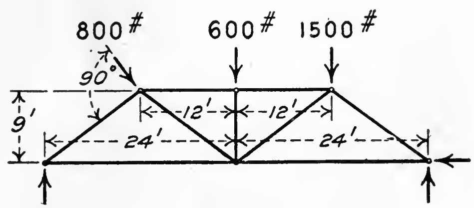


FIG. 101.

4. Figure 101 shows a roof truss hinged at the right end. Find the reactions and check. Find all the internal forces joint at a time. Construct a space diagram to the scale of 1 inch = 8 feet or 1 inch = 10 feet. Letter and solve by Bow's method to the scale of 1 inch = 500 pounds.

65. The Method of Sections.—It is often desirable to find the stress in a few members of a truss or other structure. For this purpose, the truss may be imagined to be cut at some surface into two portions. Either of these portions may be regarded as a free body in equilibrium under the action of the external forces on its side of the surface, and of the internal forces which cross the surface from the other portion. These *internal* forces which cross the imaginary surface are *external* to the portion under consideration. If there are not more than three unknown forces acting on the portion in question, these may be determined as a problem of non-concurrent forces.

Example

Figure 102 represents the truss of Fig. 99. The external reactions are known. The truss may be regarded as divided at the surface *SR*, which cuts the members *CJ*, *JI*, and *IF*. The portion to the right of *SR* will be taken as the free body This portion is shown separately in Fig. 102, II. The forces which act on the free body are the known forces of 2800 pounds, 3360 pounds, and 5700 pounds, together with the unknown forces in the members *CJ*, *JI*, and *IF*. The direction of each of these members is known; hence, the unknowns are the magnitudes of the three forces.

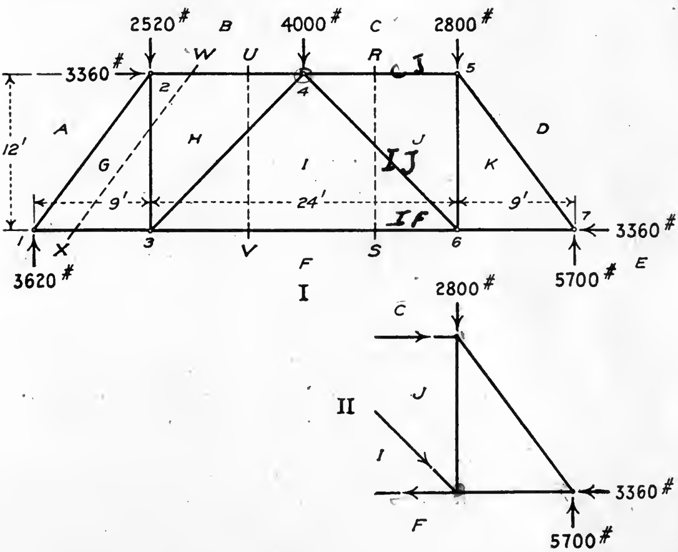


FIG. 102.

The members *JI* and *IF* intersect at joint No. 6. If moments are taken about this joint, the only unknown is the force in *CJ*.

$12 \times cj = 9 \times 5700,$ **cj = 4275 lb.**

The reaction of 5700 pounds tends to turn the portion of the truss in a counter-clockwise direction about joint No. 6. The force in *CJ* must turn in the opposite direction. Consequently *CJ* must *push* toward the right.

cj = 4275 lb. compression.

The members *JI* and *CJ* intersect at joint No. 4, Fig. 102, I. If moments are taken about this joint, the only unknown is the force in *IF*. (This origin of moments is outside the portion under consideration. If a body is in equilibrium, the sum of the moments is zero for *any origin whatever*, whether inside the body or entirely away from it.)

$$
\begin{array}{rcl}
2800 \times 12 & = & 33600 \\
3360 \times 12 & = & 40320 \\
\hline
 & & 73920 \text{ clockwise,} \\
5700 \times 21 & = & 119700 \text{ counter-clockwise} \\
\hline
if \times 12 & = & 45780,
\end{array}
$$

if = **3817 lb. tension.**

Two of the unknowns have now been found by two moment equations. The third unknown is best found by a vertical resolution,

$$ij \sin 45° = 5700 - 2800 = 2900,$$
$$ij = 2900 \text{ cosec } 45° = 2900 \times 1.4142,$$

ij = **3701 lb. compression.**

It is not at all necessary to make a separate drawing of the portion of the truss which is taken as the free body. This has been done in Fig. 102, II, in order to simplify the explanation. Generally the original space diagram, Fig. 102, I, is employed and a section line is drawn across to indicate the boundary of the portion in equilibrium.

The solution of a problem, joint at a time, may be regarded as a special case of the method of sections. A single joint is cut off as the free body and the forces are concurrent. The term, "method of sections" is applied to cases like that of Fig. 102, II, in which the forces on the free body are non-concurrent.

Problems

1. In Fig. 102, use the portion of the truss to the left of the section *RS* as the free body and solve for the forces in *CJ*, *JI*, and *IF*. Take moments about the same points as in the example above.

2. In Fig. 102, make a section *VU*, which cuts the members *BH*, *HI*, and *IF*, and find the force in the members cut without using any other internal forces. Solve first with the portion to the left of the section as the free body and check with the portion to the right of the section as the free body.

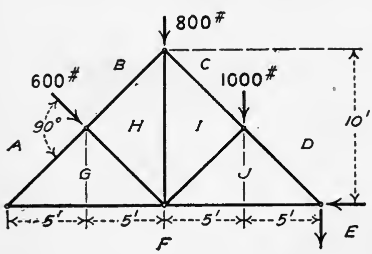


Fig. 103.

3. Make a section *WX* in Fig. 102, separating the end post *AG* as a free body. Find the force in *HG* by a vertical resolution.

4. In Problem 3 of Art. 64, calculate the external reactions, and then find *bh* and *hi* by sections without using any other internal forces.

5. In Fig. 103, find the reactions and check. Make a section through *BH*, *HG*, and *GF* and find the forces in these members, without using any other internal forces.

Could you solve Problem 5 by means of a section through *BH*, *HI*, *IJ*, and *JF*? Why?

6. Solve Problem 5 for all internal forces, joint at a time.

7. Solve Problem 5 for all internal forces by Bow's method to the scale of 1 inch = 400 pounds.

8. In Fig. 104, the loads in *AB* and *BC* are each 1200 pounds. Find the reactions. Find *bf* and *fd* by sections. Find all internal forces, joint at a time.

9. In Fig. 104, the load in *AB* is 800 pounds, and the load in *BC* is 1200 pounds. Find the reactions. Find *bf* and *fd* by sections. Check by a horizontal resolution.

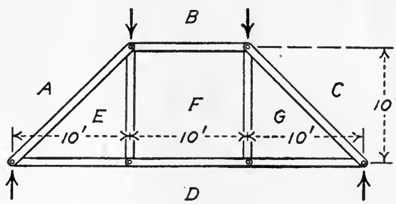


FIG. 104.

10. In Fig. 105, find the reactions in *BA*, *AF*, and *FE*. Check. Find *ci*, *ij*, and *je* by sections. Begin at the right end and solve for all forces, joint at a time. Begin at the right end and solve graphically.

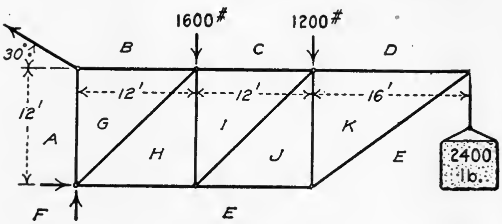


FIG. 105.

66. Jointed Frame with Non-concurrent Forces.—In the jointed frames considered, the loads have been applied at the joints; consequently the intervening members transmitted force only in the direction of their length. A member to which only two forces are applied, is called a *two-force member*. If the forces are in equilibrium, they must lie along the same straight line, and if they are applied at opposite ends of the member, the direction of each force must be in the line of the member. A member to which three forces are applied is called a *three-force member*. It is not necessary that any of these forces be in the direction of its length.

Example

Figure 106 shows two links which are connected to each other and to the supports by smooth hinges. The left link carries a load of 30 pounds 1 foot from the left end; the right link carries a load of 50 pounds at the middle. The problem is to find the reactions of the supporting hinges and the internal reaction at the hinge which connects the links.

Equilibrium equations may be written for the two links separately, or for both together as a single free body. It is generally advisable to write first the equations for the entire structure, and calculate as many external forces as possible; then write equations for parts of the structure to obtain the remainder.

The links and their support form a 3:4:5 triangle. Cos $\theta = 0.6$; $\sin \theta = 0.8$; $\cos \phi = 0.8$; $\sin \phi = 0.6$.

The vertical component of the reaction at the left hinge is V_1. The vertical component of the reaction at the right hinge is V_2. The horizontal component at the left hinge is H_1. The horizontal component at the right hinge is H_2.

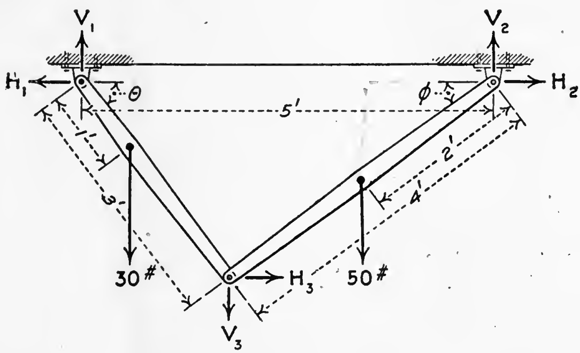


FIG. 106.

Taking moments about the left hinge,

$$5V_2 = 30 \times 0.6 + 50 \times 3.4,$$

$$\begin{aligned} 30 \times 0.6 &= 18 \\ 50 \times 3.4 &= 170 \\ \hline 5V_2 &= 188 \end{aligned} \qquad \mathbf{V_2 = 37.6\ lb}$$

Taking moments about the right hinge,

$$\begin{aligned} 50 \times 1.6 &= 80 \\ 30 \times 4.4 &= 132 \\ \hline 5V_1 &= 212 \end{aligned} \qquad \mathbf{V_1 = 42.4\ lb.}$$

There are still two external forces, H_1 and H_2. By horizontal resolution these are known to be equal and opposite. All the possible equations for a problem of non-concurrent forces have now been used and there is still one unknown to be found. The vertical reactions depend entirely upon the horizontal distances of the lines of the loads from the two supports and are independent of the length of the links and the kind of connections between them.

To find H_2, the right link may be taken as the free body and a moment equation written about the hinge which connects the two links.

$$\begin{aligned} 37.6 \times 3.2 &= 120.32 \text{ counter-clockwise,} \\ 50 \quad \times 1.6 &= 80.00 \text{ clockwise,} \\ \hline 2.4H_2 &= 40.32, \end{aligned} \qquad \mathbf{H_2 = 16.8\ lb.}$$

The horizontal reaction at the left hinge is 16.8 pounds toward the left.

In Fig. 106, H_3 is the horizontal component of the force from the right link to the left link, and V_3 is the vertical component. Equal and opposite forces act from the left link to the right link.

Using the left link as the free body and taking vertical resolutions,

$$V_1 + V_3 = 30 \text{ lb.}$$
$$V_3 = 30 - 42.4, \quad \mathbf{V_3 = 12.4\ lb.\ downward.}$$

If V_4 is the vertical component of the reaction of the left link on the right link, a vertical resolution with the right link as the free body gives an equal force in the opposite direction.

By horizontal resolutions, with either link as the free body, $\mathbf{H_3 = 16.8\ lb.}$ Both H_3 and V_3 may be checked by moments about the left hinge with the left link as the free body.

Problems

1. In Fig. 107, find the direction and magnitude of the reactions at the supporting hinges, and the direction and magnitude of the force from the inclined member to the horizontal member at the hinge which connects them.

Ans. $H_1 = H_2 = H_3 = 163.92$ lb.;
$V_1 = 25.36$ lb.; $V_2 = 134.65$ lb.; $V_3 = 34.64$ lb.;
$R_1 = 165.9$ lb. left 8° 48′ up;
$R_2 = 212.1$ lb. right 39° 24′ up;
$R_3 = 167.5$ lb. right 11° 32′ up.

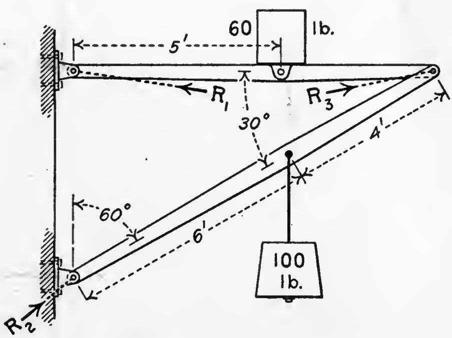


FIG. 107.

2. In Problem 1, find the location of the resultant of 100 pounds and 60 pounds by moments. Then, with the direction of R_1 known from the algebraic solution, find the direction of R_2 by means of the direction condition of equilibrium of three forces. Finally, draw the force triangle and get the magnitudes of R_1 and R_2.

3. In Fig. 108, find the direction and magnitude of the reaction at each hinge.

A problem in which the members of a connected system are subjected to non-concurrent forces may be changed into a problem of the ordinary type by considering the forces between the joints as replaced by forces at the joints. In Fig. 109, which is the same as Fig. 106, the load of 50 pounds at the middle of the

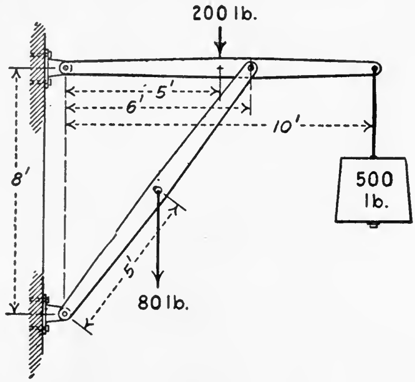


FIG. 108.

right link has been replaced by a load of 25 pounds at each end of the member; and the load of 30 pounds at the third point of the left link has been replaced by 20 pounds at the left end and 10

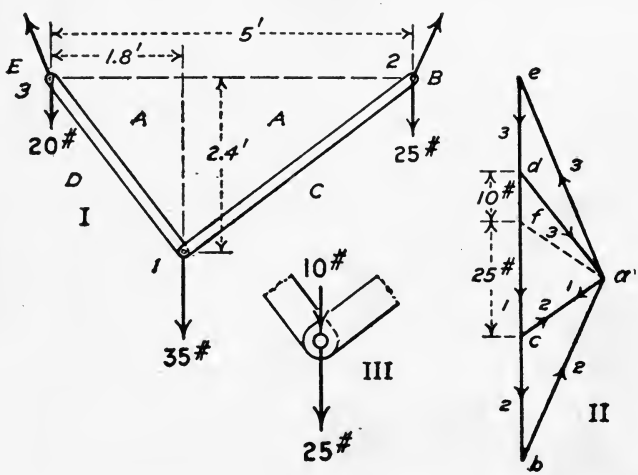


FIG. 109.

pounds at the right end. The load of 10 pounds at the right end of the left link and the load of 25 pounds at the left end of the right link together are equivalent to 35 pounds. As far as the external forces are concerned, the problem of Fig. 109, I, is equivalent to the problem of Fig. 106.

In Fig. 109, I, single reactions have been drawn at each support, instead of the horizontal and vertical components. These components, however, might have been used.

Figure 109, II, shows the graphical solution. At the lower joint, there are two unknowns in the direction of the links. Beginning with the known load of 35 pounds vertically downward, draw the force triangle *dca*. At the right hinge, the unknowns are the direction and magnitude of the hinge reaction. Lay off the line *cb*, 25 units in length, in a vertical direction. The closing line *ba* gives the direction and magnitude of the hinge reaction. The horizontal component of the hinge reaction is the force H_2 of Fig. 106, and the vertical component is the force V_2. The left reaction *de* is found in a similar manner.

To find the direction and magnitude of the reaction at the lower hinge, the load of 25 pounds is regarded as acting on the right link, and the load of 10 as acting on the left link. The pin may be regarded as replaced by a short link and an additional letter inserted in the space diagram. On the force diagram measure 10 units downward from *d* to the point *f*. The broken line *af* is the reaction at the pin.

Problems

4. In Problem 1, transfer the vertical loads to the joints and solve graphically.

5. In Problem 3, Fig. 108, transfer the vertical loads to the joints and solve graphically.

67. Summary.—The magnitude and direction of the resultant of a set of non-concurrent, coplanar forces is the vector sum of the forces. If the forces are all parallel, the vector sum is the algebraic sum. In these respects, non-concurrent forces and concurrent forces are alike.

The *location* of the resultant is found from the condition that the moment of the resultant about any point is equal to the moment of the separate forces about that point.

For equilibrium, the force polygon must close, and the sum of the moments about any point must be zero.

For the algebraic solution of a problem of the equilibrium of non-concurrent, coplanar forces, three independent equations are written. These may be:

1. One moment equation and two resolution equations.
2. Two moment equations and one resolution equation.
3. Three moment equations.

When two moment and one resolution equation are written, the resolution must be not taken perendicular to the line which joins the two origins of moment. When three moment equations are written, the three origins of moment must not lie in the same straight line.

When the forces are all parallel, there can be only two unknowns and two independent equations. One of these equations must be a moment equation.

The direction condition of equilibrium for three forces requires that the lines of action of the forces must meet at a point. The direction condition for four forces is that the resultant of two of the forces must pass through the intersection of the other two. A direction condition may replace a moment equation in the solution of a problem of equilibrium. The direction conditions of equilibrium are especially useful in problems in which angles are to be found.

A resolution perpendicular to the direction of an unknown force eliminates that force. A moment equation with respect to a point in the line of action of an unknown force eliminates that force. It is often desirable to take moments about the point of intersection of two unknown forces, or about a point in the line of action of a force whose direction and magnitude are both unknown.

In a connected system, the entire system is first treated as the free body and as many of the reactions as possible are calculated. The parts of the system are then treated as free bodies. When the free body is a single joint, the forces are concurrent. The method of sections divides the system into two portions. Either portion may be treated as the free body in equilibrium. The forces which act on the portion are the external forces on its side of the section, and the internal forces in the members which are cut by the section.

68. Miscellaneous Problems

1. Two bodies, one of which weighs 20 pounds and the other 50 pounds, are attached to the ends of a rope which runs over a smooth pulley. The 50-pound mass is in contact with a smooth plane which makes an angle of 15 degrees with the horizontal. Find the position of equilibrium, the reaction of the plane, and the direction and magnitude of the resultant force on the pulley.

2. In Problem 1, what is the maximum angle of the plane with the horizontal in order that equilibrium may be possible?

3. In Problem 1, both bodies are in contact with the plane. The perpendicular distance from the pulley to the plane is 5 feet. Neglecting the dimensions of the pulley, find the minimum length of rope for possible equilibrium.

4. A 20-pound mass is supported by two cords. One cord is 5 feet long and has the upper end attached to a point A. The second cord runs over a smooth pulley at the same level as the point A, and 5 feet therefrom, and carries a load of 12 pounds. Find the position of equilibrium, the tension in the first cord, and the resultant force on the pulley. There are two possible solutions of the equations. What change must be made in the construction of the apparatus in order that both solutions may be mechanically possible?

5. Solve Problem 4 if the pulley is 5 feet from A along a line which makes an angle of 10 degrees with the horizontal. Write moments about A and solve by trial and error.

Ans. The cord from A makes an angle of 29° 18′ with the vertical.

6. A bar 10 feet long, weighing 60 pounds, with its center of mass 6 feet from the left end, is hinged at the left end and supported by means of a rope at the right end. The right end is elevated 20 degrees above the horizontal. The rope at the right end makes an angle of 35 degrees to the left of the vertical. Find the tension in the rope and the direction and magnitude of the hinge reaction by means of one moment equation and two resolution equations. Check by a second moment equation.

7. Solve Problem 6 by means of the direction condition of equilibrium. Compare with the results of Problem 6.

8. A box is 5 feet long horizontally and 3 feet high. It weighs 240 pounds and its center of mass is at the center. A force applied at the upper right edge tips the box about the lower left edge. The force makes an angle of 25 degrees to the left of the vertical. If the friction is sufficient to prevent sliding, what force will be required to start the box?

Ans. 103.4 lb.

9. Solve Problem 8 graphically by means of the direction condition of equilibrium.

10. In Problem 8, what is the direction and magnitude of the minimum force at the upper right edge which will start to tip the box?

11. The box of Problem 8 is turned through an angle of 20 degrees. Find the direction and magnitude of the smallest force at the upper right corner which will hold it in this position?

12. A ladder 20 feet long, weighing 50 pounds, with its center of mass 8 feet from the lower end, stands on a smooth horizontal floor and leans against a smooth, vertical wall. It is held from slipping by a horizontal force at the floor. Find this force and the reaction at the floor and at the wall when the ladder makes an angle of 25 degrees with the vertical.

13. The ladder of Problem 12 stands on a smooth horizontal floor and leans against the edge of a wall 15 feet in height. It is held from slipping by a horizontal force at the bottom. Solve for the unknowns when the ladder makes an angle of 30 degrees with the horizontal.

14. Solve Problem 12 if the horizontal force at the bottom of the ladder is 16 pounds and the position is unknown.

15. Solve Problem 12 if the horizontal force is 15 pounds and is applied 2 feet from the bottom of the ladder. Check your results by means of the direction condition of equilibrium.

16. A beam 20 feet long, weighing 80 pounds, with its center of mass at the middle, has its left end on a smooth plane which makes an angle of 60 degrees to the left of the vertical, and its right end on a smooth plane which makes an angle of 70 degrees to the right of the vertical. The beam carries a load of 60 pounds 5 feet from the left end, and a load of 100 pounds 4 feet from the right end. Find the reactions of the planes and the angle which the beam makes with the horizontal.

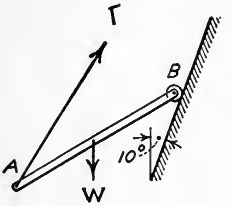


FIG. 110.

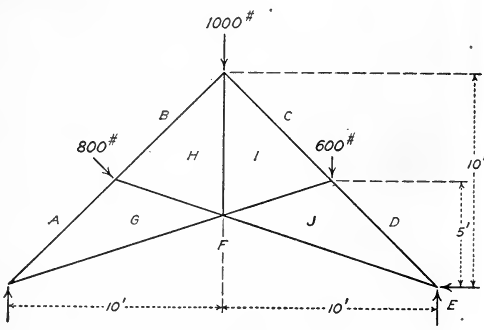


FIG. 111.

17. A bar 4 feet in length, weighing 50 pounds, with its center of mass at the middle, has one end against a smooth vertical wall. The other end is attached to a rope 5 feet in length which is fastened to a point in the wall. Find the position of equilibrium, the reaction of the wall, and the tension in the rope.

18. The bar AB, Fig. 110, has its center of mass at the middle of its length. The end B is against a smooth wall which makes an angle of 10 degrees with the vertical. The end A is supported by a rope. Find the direction of the rope, the tension in it, and the reaction of the plane.

19. In Fig. 111, find all reactions and check. Find bh, hg, and gf by sections. Find all forces joint at a time. With the reactions known, solve by Bow's method.

CHAPTER VI

COUPLES

69. Moment of a Couple.—In Fig. 112, **P** and **Q** are two forces in opposite directions at a distance a apart. Their equilibrant is the force **Q** − **P** at a distance x from **Q**. The distance x is given by the equation,

$$(Q - P)x = Pa \qquad (1)$$

If $Q = P$, $Q - P = 0$, $x =$ infinity.

Two equal and opposite forces acting on a body form a *couple.* The forces of a couple have a moment about any point and tend to produce *rotation* about any point. Since the forces are equal, their resultant is zero. The forces of a couple do not produce *translation.*

If the magnitude of each force is P and the distance between the forces is a, the moment is Pa. The moment is the same with

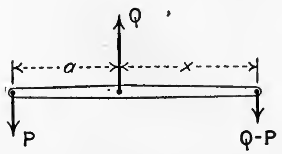


FIG. 112

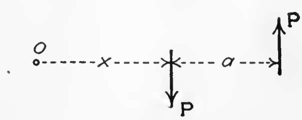


FIG. 113.

respect to any point in the plane of the forces. This statement may be proved by Fig. 113. The point O, at a distance x from the downward force of the couple, may be taken as the origin of moments. The sum of the moments of the two forces about this origin is

$$M = -Px + P(x + a) = Pa \qquad (2)$$

The moment of the couple is the same, no matter what point is taken as the origin. *The moment of a couple is the product of either force multiplied by the distance between the lines of action of the two forces.*

70. Equivalent Couples.—Two *forces* are equivalent if either force may be balanced by a third force so that no *translation* is produced in the body upon which the forces act. Two *couples* are equivalent if either couple may be balanced by a third couple so that no *rotation* is produced in the body upon which the couples act. These definitions may be stated briefly: *Forces which are balanced by the same force are equivalent. Couples which are balanced by the same couple are equivalent.*

In order that two forces may produce equilibrium, the forces must be equal, opposite, and along the same line. In order that two couples may be in equilibrium, the magnitude of their moments must be equal, the direction of rotation must be opposite, and the couples must lie in the same plane or in parallel planes.

It will now be proved that a given couple may be balanced by a second couple in the *same plane*, provided the moments are equal and opposite. If the moments are equal and opposite, the forces may have any magnitude and may be placed at any position in the plane. A couple may be made of an upward

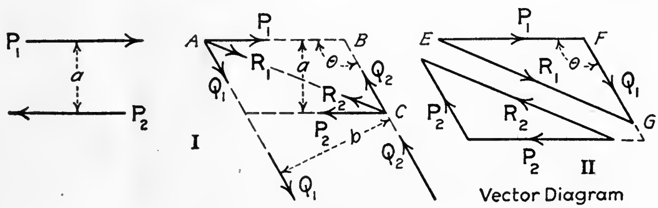


Fig. 114.

force of 6 pounds and a downward force of 6 pounds at a distance of 4 feet to the right of the first force. The moment of this couple is 24 foot-pounds clockwise. Any counter-clockwise couple of 24 foot-pounds in the same plane will produce equilibrium. This second couple may be made of two horizontal forces of 24 pounds at a distance of 1 foot apart, with the upper force toward the left, or it may be made of two forces of 12 pounds at a distance of 2 feet apart, or of two forces of 8 pounds at a distance of 3 feet, or of any other combination whose moment is 24 foot-pounds counter-clockwise.

In Fig. 114, $\mathbf{P}_1$ and $\mathbf{P}_2$ are equal forces of magnitude P at a distance a apart. These forces form a clockwise couple of magnitude Pa. A second pair of forces, $\mathbf{Q}_1$ and $\mathbf{Q}_2$, each of magnitude Q, at a distance b apart, form a counter-clockwise couple of moment Qb. It will be proved that these couples produce equilibrium if the magnitude of the moment Pa is equal to the magnitude of the moment Qb.

Extend the lines of the forces until the line of the force $\mathbf{P}_1$ intersects the line of the force $\mathbf{Q}_1$, and the line of the force $\mathbf{P}_2$ intersects the line of the force $\mathbf{Q}_2$. The resultant of the forces $\mathbf{P}_1$ and $\mathbf{Q}_1$ at A is a force $\mathbf{R}_1$. The direction and magnitude of the force $\mathbf{R}_1$ are given by the vector diagram, E, F, G, of Fig. 114, II. The resultant of $\mathbf{P}_2$ and $\mathbf{Q}_2$ at C is a force $\mathbf{R}_2$. The magnitude and direction of $\mathbf{R}_2$ are given by the lower force triangle of Fig. 114, II. Since $\mathbf{P}_1$ is equal and opposite to $\mathbf{P}_2$, and $\mathbf{Q}_1$ is equal and opposite to $\mathbf{Q}_2$, it is evident from Fig. 114, II, that $\mathbf{R}_1$ is equal and opposite to $\mathbf{R}_2$. If the resultant $\mathbf{R}_1$ through A of the space diagram falls on the same line as the resultant $\mathbf{R}_2$ through C, the forces will balance and the couples will be in equilibrium. This condition is satisfied if the line of $\mathbf{R}_1$ passes through the point C.

In the space triangle ABC, and the force triangle EFG, the angle at B is equal to the angle at F. If this angle is represented by θ;

$$AB \sin \theta = b, \tag{1}$$

$$BC \sin \theta = a, \tag{2}$$

$$\frac{AB}{BC} = \frac{b}{a}. \tag{3}$$

The triangles ABC and EFG are similar if

$$\frac{AB}{BC} = \frac{EF}{FG}, \tag{4}$$

and the resultant $\mathbf{R}_1$ on the space diagram falls on the line AC. Substituting from Equation (3)

$$\frac{b}{a} = \frac{EF}{FG} = \frac{P}{Q}, \tag{5}$$

$$Pa = Qb \tag{6}$$

Equation (6) states that the moments of the two couples are equal, when ABC and EFG are similar.

Since the couple Pa may be balanced by any other couple of equal moment and opposite direction in the same plane, it follows *that any two couples in the same plane are equivalent if moments are equal in magnitude and sign.*

If the moment Pa of Fig. 114 does not equal the moment Qb, the resultant $\mathbf{R}_1$ will still be equal and opposite to the resultant $\mathbf{R}_2$. The two resultants will not, however, lie on the same line but will form a new couple of moment $Pa - Qb$.

Problems

1. A couple is made up of a horizontal force of 12 pounds toward the right and an equal force toward the left at a distance of 3 inches below the first force. A second couple is made up of a force of 8 pounds upward at an angle of 30 degrees to the right of the vertical and an equal force in the opposite direction at a distance of 5 inches, so placed that the moment is counter-clockwise. Draw a space diagram similar to Fig. 114 to the scale of 1 inch = 1 inch. Construct the force diagram similar to *EFG* of Fig. 114 to the scale of 1 inch = 4 pounds, and find $\mathbf{R}_1$. Through *A* and *C* draw lines parallel to $\mathbf{R}_1$. Measure the distance between these lines and multiply by $\mathbf{R}_1$. Compare the product with the difference of the two moments.

2. Solve Problem 1 if the forces of the second couple are so placed that its moment is clockwise.

3. A clockwise couple is made up of two horizontal forces of 10 pounds each at a distance of 6 inches from each other. A counter-clockwise couple in the same plane is made up of two forces of 15 pounds each at a distance of 4 inches apart. Find their resultant graphically.

It will be shown in Art. 106 that couples in *parallel* planes which are equal in magnitude and have the same signs are equivalent.

71. Algebraic Addition of Couples.—Couples in the same plane may be added graphically by methods of Art. 70. It will

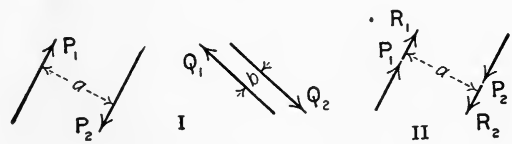


FIG. 115.

now be proved that the combined moment of two couples in the same plane is the algebraic sum of the moments of the couples. If *Pa* is one couple and *Qb* is another couple in the same plane, the resultant moment is $Pa + Qb$ when the moments are in the same direction, and the resultant moment is $Pa - Qb$ when the moments are in opposite directions.

There are two couples in Fig. 115. If $\frac{Qb}{a} = R$, the couple *Qb* may be replaced by a second couple *Ra*. The forces of the second couple are $\mathbf{R}_1$ and $\mathbf{R}_2$, each of magnitude *R*. According to Art. 70, this couple *Ra* may be placed anywhere in the plane of the original couples. In Fig. 115, II, the force $\mathbf{R}_1$ is placed in the line of the force $\mathbf{P}_1$, and the force $\mathbf{R}_2$ is placed in the line of the force $\mathbf{P}_2$. The resultant force in each of these lines is now $\mathbf{P} + \mathbf{R}$ and

$$M = (P + R)a = Pa + Ra = Pa + Qb$$

This equation proves the proposition. Any number of couples may be added in the same way.

Since the moment of a couple is the same about any point in its plane, it might be regarded as self-evident that the combined moment of several couples is the algebraic sum of the separate moments. The proof here given is, however, more satisfactory to most readers, than this brief statement.

72. Equilibrium by Couples.—In many problems of equilibrium, the forces may be grouped to form two equal and opposite

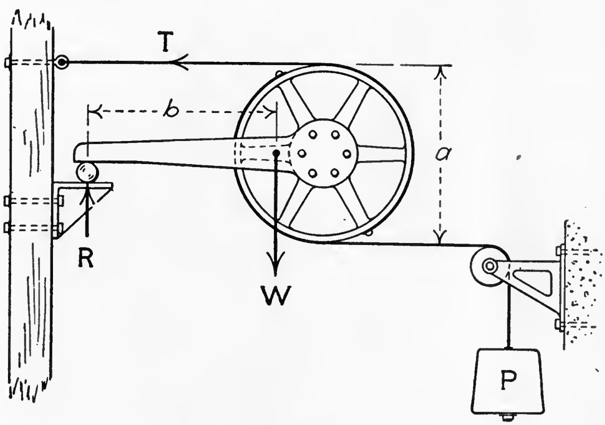


FIG. 116.

couples in the same plane. Figure 116 represents an arrangement for demonstrating the equilibrium of couples. A rigid body is made up of a bar and a wheel fastened together. The bar is supported by a small cylinder which rolls on a plane surface. The reaction of the small cylinder is vertical. This reaction together with the weight of the system forms one couple. Two cords are passed partly around the wheel and fastened to it. One cord runs horizontally toward the left and is attached to a post. The second cord runs horizontally toward the right, passes over a smooth pulley, and supports a mass of P *pounds.* $P = T$ and $R = W$ For equilibrium

$$Pa = Wb$$

The plane supporting the cylinder may be a platform scale.

If the scale poise is set to equal the load W and the weight of the small cylinder, the beam will be in balance when the cords are parallel.

Figure 117 shows a beam with loads **P** and **Q** near the ends.

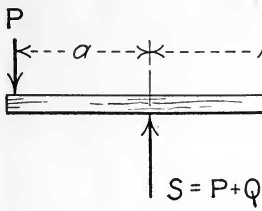


FIG. 117.

Neglecting the weight of the beam, the reaction **S** = **P** + **Q**. The load **P** at the left end together with an equal amount of the reaction at the middle forms a couple of moment Pa. The load **Q** at the right end, together with the remainder of the reaction,

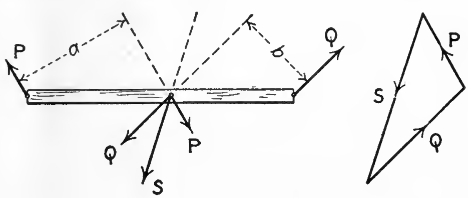


FIG. 118.

forms an opposite couple of moment Qb. For equilibrium these couples are equal.

In Fig. 118, the two forces **P** and **Q** are not parallel. The equilibrant **S** may be resolved into two components, which are equal and opposite to **P** and **Q**, respectively. The beam is then subjected to two couples of moments Pa and Qb.

Problems

1. What are the two equal couples in Fig. 96 of Art. 63?

2. What are the two equal couples of Fig. 88?

3. In Fig. 89, when AC and BD are parallel, what are the two equivalent couples?

73. Reduction of a Force and a Couple to a Single Force.—In Fig. 119, I, there is a single force **P**, and a counter-clockwise couple in the same plane made up of two equal forces $\mathbf{Q}_1$ and

$\mathbf{Q}_2$ at a distance b apart. The moment of this couple is Qb. If $Pa = Qb$, this couple may be replaced by a couple made up of two equal forces $\mathbf{P}_1$ and $\mathbf{P}_2$ at a distance a apart. By Art. 70 these forces may be placed anywhere in the plane of the couple. The force $\mathbf{P}_1$ may be placed in the line of action of the single force P, and the force $\mathbf{P}_2$ at a distance a from that line on the side which makes the moment counter-clockwise. The force $\mathbf{P}_1$, Fig. 119, II, balances the force $\mathbf{P}$. The remaining force $\mathbf{P}_2$, at a distance a from the position of the force $\mathbf{P}$, replaces the force and the couple.

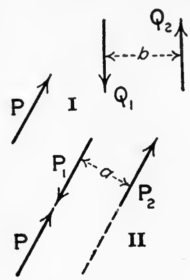


FIG. 119.

A force and a couple in the same plane are equivalent to a single force. The direction and magnitude of the single force are the same as those of the line of action of the original force. Its distance from the line of action of the original force is such that its moment about any point in that line is equal in magnitude and sign to the moment of the original couple.

Example

A vertical force of 20 pounds upward is applied at the point $x = 4$ ft. A horizontal force of 5 pounds toward the right is applied at $y = 2$ ft. and an equal horizontal force toward the left is applied at $y = 12$ ft. Find the location of the single force which is equivalent to this force and couple.

The moment of the couple is 50 foot-pounds.

$$\frac{50}{20} = 2.5 \text{ ft.}$$

The single force is 20 pounds upward at a distance of 2.5 feet from the vertical line through the point $x = 4$ ft. Since the couple is counter-clockwise, the distance of 2.5 feet must be measured toward the right from the line $x = 4$ ft. The resultant force lies in the line $x = 6.5$ ft.

Problems

1. A force of 12 pounds, along the line $x = 2$ ft., is combined with a couple made up of a horizontal force of 8 pounds toward the right, along the line $y = 1$ ft., and an equal and opposite force, along the line $y = 13$ ft. Find the location of the single force which is equivalent to the force and the couple.

2. Solve Problem 1 graphically to the scale of 1 inch = 4 feet, and 1 inch = 8 pounds. Through the intersection of the line of action of the vertical force with the line of action of one of the horizontal forces draw a line parallel to the direction of their resultant. Through the intersection of this line with the line of the third force on the space diagram, draw a line parallel to the direction of the resultant of all three forces.

3. Find the single force which can be substituted for a horizontal force of 16 pounds directed toward the right and a clockwise couple of 48 foot-pounds.

Ans. 16 pounds towards the right through a point 3 ft. above the original force.

4. An upward vertical force of 8 pounds acts along the Y-axis, and a second upward vertical force of 4 pounds acts along the line $x = 6$ ft. A couple of 60 foot-pounds clockwise and a couple of 18 foot-pounds counter-clockwise act on the body in the same plane with these forces. Find the single force which will replace all of these.

74. Resolution of a Force into a Force and a Couple.—Figure 120 shows a single force **P**. It is desired to replace this *force* by the *force* and a couple of moment Pa. In Fig. 120, C is a point at a distance a from the line of action of the force **P**. At C are applied two opposite forces, $\mathbf{P}_1$ and $\mathbf{P}_2$, each of which is equal in magnitude to the force **P**, and along a parallel line. Since these forces balance each other, they have no effect upon the equilibrium of the body upon which they act. The force $\mathbf{P}_1$ and the force **P** form a couple of moment Pa. The force $\mathbf{P}_2$ which is equal and parallel to the original force **P**, stands alone as the single force required.

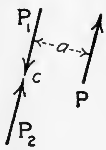


Fig. 120.

A single force may be replaced by an equal force in the same direction through any point in its plane, and a couple, the moment of which is the same in magnitude and direction as the moment of the original force about that point.

Problems

1. A force of 12 pounds upward acts at the point $x = 3$. Replace this force by an equal force through the origin and a couple.

Ans. 12 pounds upward through the origin and a counter-clockwise couple of 36 units.

2. A force of 16 pounds at an angle of 45 degrees to the right of the vertical upward acts at the point $x = 3$ ft., $y = 5$ ft. Replace by a force through the origin and a couple.

Ans. 16 pounds through the origin at an angle of 45 degrees to the right of the vertical, and a counter-clockwise couple of 22.62 foot-pounds.

3. A force of 20 pounds at an angle of 25 degrees to the right of the vertical is applied at the point $x = 2$ ft., $y = 3$ ft. and a force of 15 pounds at an angle of 60 degrees to the right of the vertical is applied at the point $x = 2$ ft., $y = 8$ ft. Replace these by a single force at the origin and a single couple.

Replace each force by a single force at the origin and a single couple. Add the two couples and find the resultant of the two forces.

The principles of this article afford a method of finding the resultant of a set of non-concurrent, coplanar forces. Let $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, etc be a set of forces in the same plane. Each force may be replaced by an equal force in the same direction applied at some convenient point and a couple. Since all the forces are now applied at the same point, they may be treated as concurrent and their resultant found by means of the force polygon or calculated by the methods of Art. 42. The couples may be added algebraically. Their sum is a single couple.

The resultant of a set of non-concurrent forces in the same plane is equivalent to a single force and a single couple. Since a force and a couple in the same plane may be reduced to a single force by the methods of Art. 73, it follows that, in general, the resultant of a set of non-concurrent, coplanar forces is a single force. It sometimes happens that the resultant force is zero, while the resultant couple is not zero. This resultant couple can not be reduced to a single force.

A set of non-concurrent, coplanar forces may always be reduced to a *single force* or to a *single couple.*

While the steps of the proof are different, the results of this article are practically the same as the method of finding the resultant, which is given in Art. 60.

75. Summary.—Two equal, opposite forces form a couple. The moment of a couple is the product of either force multiplied by the distance between them. The moment of a couple is the same with respect to any origin.

Two couples in the same plane are equivalent if their moments are equal in magnitude and sign.

The combined moment of several couples in the same plane is the algebraic sum of the separate moments.

A force and a couple in the same plane may be replaced by a single force which is equal in magnitude and direction to the original force, and is so located that its moment about any point in the line of the original force is equal to the moment of the couple.

A single force may be replaced by an equal force in the same direction through any point in its plane, and a couple whose moment is equivalent to the moment of the force about that point.

A set of non-concurrent, coplanar forces may be replaced by a single force and a single couple. These may be reduced to a single force, except when the resultant force is zero. In this case, the final resultant is a couple.

CHAPTER VII

GRAPHICS OF NON-CONCURRENT FORCES

76. Resultant of Parallel Forces.—In Art. 55, a method was given for finding the resultant of parallel forces in the same direction. In Fig. 76, the rigid bar was assumed to be connected to two ropes or hinged rods, and the equilibrant was found as a problem of connected bodies with concurrent forces. This proof was given because it is concrete and because it fits the methods of the preceding chapter. A more abstract proof is generally used. This will now be given.

Figure 121, I, shows two parallel forces **P** and **Q** which are applied to the ends of a rigid bar. The force **P** at the left end is

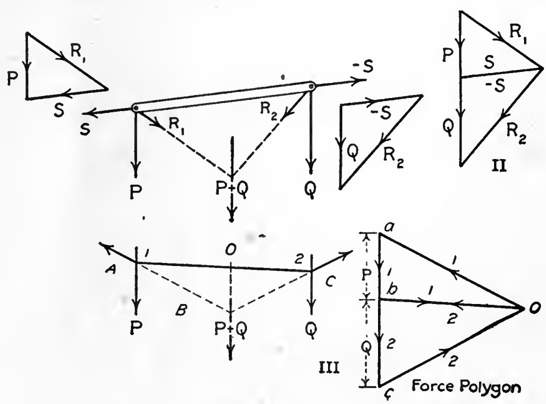


FIG. 121.

resolved into two components $\mathbf{R}_1$ and **S**, which are not at right angles to each other. These are shown in the triangle at the left of the bar. The component **S** is parallel to the bar. The force **Q** at the right end is resolved into two components $-\mathbf{S}$ and $\mathbf{R}_2$. The component $-\mathbf{S}$ is equal and opposite to thec omponent **S** at the left end. On the space diagram, it is seen that the force **S** balances the force $-\mathbf{S}$ and the components $\mathbf{R}_1$ and $\mathbf{R}_2$ remain to act on the bar. The resultant of $\mathbf{R}_1$ and $\mathbf{R}_2$ must pass through the intersection of the broken lines of the space diagram, which are drawn parallel to the corresponding forces of the force triangles. Figure 121, II shows the two force triangles combined, as they are generally drawn. The components **S** and **−S** fall on the same line. The resultant of $\mathbf{R}_1$ and $\mathbf{R}_2$ is the force $\mathbf{P} + \mathbf{Q}$, in the direction of the original forces.

Figure 121, III, shows the usual method of construction for the position of the resultant. The parallel forces are **P** and **Q**. It is not necessary to draw the body upon which they act, or to consider the direction of the line which joins their point of application. The force diagram is first drawn. The distances $ab = P$ and $bc = Q$ are laid off on a vertical line. Then a point O is selected to one side of this line, and connected to the points a, b, and c by lines ao, bo, and co. The figure thus obtained is the same as the force diagram of Fig. 121, II. A point No. 1 is chosen on the line of the force **P** and a line is drawn through it parallel to ao and a second line parallel to bo. The line parallel to bo is extended till it intersects the line of the force **Q** at the point No. 2. Through point No. 2, a line is drawn parallel to the line co of the force diagram. The intersection of the line parallel to ao with the line parallel to co gives a point on the line of the resultant. The resultant is equal and parallel to ac of the force diagram.

The space diagram of Fig. 121, III is called a *string polygon*, or *funicular polygon*. The solid lines of the drawing give the directions which three connected strings could take when supporting the loads **P** and **Q** at a definite distance apart. The diagram is lettered by Bow's method to correspond with the force diagram of the figure. (The line $P + Q$ does not represent a force, but only the position of the resultant.) The arrows show the directions required for equilibrium.

The point O of the force polygon of Fig. 121, III, is called the *pole*. The lines ao, bo, and co are *rays*. The position of the pole may be chosen at any point. It should be so selected that lines parallel to the rays will intersect at convenient positions on the string polygon.

Problems

1. Two vertical loads of 15 pounds and 20 pounds are 6 inches apart. Draw a force polygon to the scale of 1 inch = 5 pounds and put the pole 5 inches to the right of the load line. Draw the string polygon to the scale of 1 inch = 1 inch and find the distance of the resultant from the loads. Check by moments. Change the position of the pole and solve again.

2. Solve Problem 1 with the pole to the left of the load line. Explain the result.

3. Draw the force polygon and the funicular polygon for three vertical forces: 13 pounds downward at 0 feet, 10 pounds downward at 4 feet, and 18 pounds downward at 12 feet. Use scale of 1 inch = 5 pounds and 1 inch = 2 feet. Compare with Fig. 122 which is drawn for a different loading. Locate the resultant force and check by moments.

4. Three weights of 10 pounds at 0 feet, 15 pounds at 4 feet, and 10 pounds at 8 feet, are connected by cords and supported by two cords. The cords supporting the first and last weights make angles of 45 degrees with the horizontal. Find the direction and length of the cords connecting the weights.

5. Given three vertical loads: 12 pounds at 0 feet, 16 pounds at 5 feet, and 20 pounds at 8 feet. Find the location of the resultant graphically and check by moments. If the cord supporting the 12 pounds makes an angle of 60 degrees to the left of the vertical, and the cord supporting the 20 pounds maks an angle 50 degrees to the right of the vertical, what is the direction and length of the connecting cords?

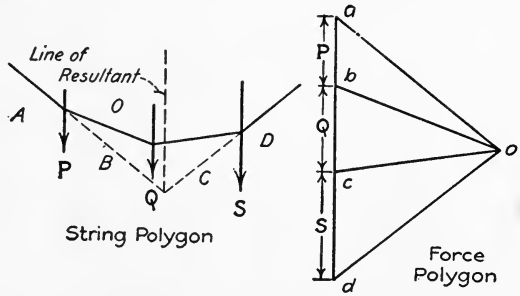


FIG. 122.

77. Resultant of Non-parallel Forces.—Art. 59 gives a method of finding the resultant of non-parallel forces graphically. It frequently happens that some of the forces are so nearly parallel that this method requires an extremely large space diagram to get the intersections. In such a case the results are likely to be inaccurate. In most cases of non-parallel forces, especially when there are more than three forces, it is advisable to use the method of Art. 76. This differs from the method of Art. 59 only in the fact that the first and last forces are resolved into components, with one component of the first force equal and opposite to one component of the last force, and acting along the same line. By properly choosing the location of the pole in the force diagram, the rays may be given such directions that the forces which they represent will make large angles with the directions of the applied loads on the space diagram. The intersections will then be found accurately.

Example

A rectangular board in a vertical plane is 4 feet wide horizontally and 3 feet high. The board weighs 20 pounds and its center of mass is at the center. A cord, 30 degrees to the left of the vertical downward is applied at the lower left corner and exerts a pull of 12 pounds. A cord at the lower right corner makes an angle of 15 degrees to the right of the vertical downward and exerts a pull of 15 pounds. A cord at 45 degrees to the right of the vertical downward is attached to the upper right corner and exerts a pull 16 pounds. The board is held in equilibrium by a single cord. Find the direction and location of this cord and the force which it exerts.

Figure 123 gives the solution. The original drawing was made to the scale of 1 inch = 1 foot on the space diagram and 1 inch = 10 pounds on the force diagram. The space diagram, showing the position and direction of the known forces, was first drawn. Next, the load line *abcde* of the force polygon was laid off. The pole was taken 4 inches from the middle of the vertical line *bc*.

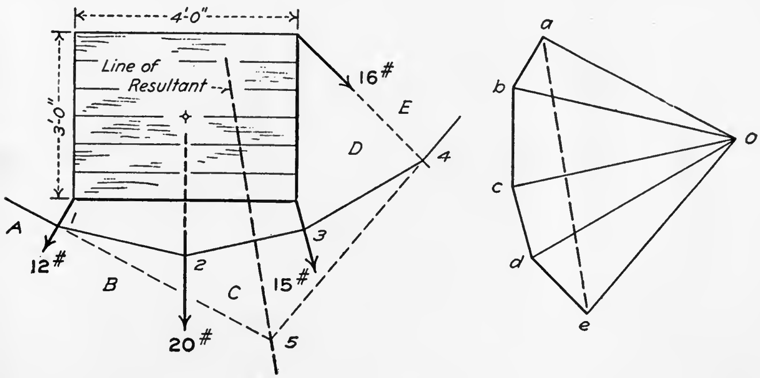


FIG. 123.

This position makes the ray *ao* nearly normal to the direction of the force *ab*, and the ray *eo* nearly normal to the force *de*, and makes large angles between the rays and the forces which they intersect. Next, a point No. 1 was chosen in the line of the 12-pound force on the space diagram. Through point No. 1, a line was drawn toward the left parallel to the ray *ao*, and a second line toward the right parallel to the ray *bo*. This second line intersects the line of the 20-pound force at point No. 2. This operation was continued for all the rays. Finally, the first line through point No. 1 was extended to the right and the last line through point No. 4 was extended to the left. The intersection of these lines is point No. 5, which lies on the line of the resultant (or the equilibrant) of all these forces. The resultant of all these forces is the line *ae* of the force polygon. The broken line through point No. 5, drawn parallel to *ae*, gives the line of action of the resultant. A single cord attached to the board at any point on this line may hold it in equilibrium.

Problems

1. Solve the Example above graphically, choosing the pole 3 inches from *c* on the horizontal line through *c*.

2. The following forces act on a rigid body: 20 pounds 18 degrees to the left of the vertical downward, applied at the point (−4, 2); 16 pounds vertically downward, applied at the point (−1, 0); 18 pounds 25 degrees to the right of the vertical downward, applied at the point (2, 1); and 24 pounds 40 degrees to the right of the vertical downward, applied at the point (4, 0). Find the magnitude, direction, and location of the resultant graphically.

3. Solve the Example of Art. 60 by the graphical method.

78. Parallel Reactions.—When a body is subjected to a set of forces in the same plane, it may be held in equilibrium by a single force in that plane, except when the resultant is a couple. This equilibrant is equal and opposite to the resultant of all the forces and lies in the same line. The line of the resultant in the space diagrams of the problems of Art. 77 gives the location of a series of points at which the body may be supported by *a single smooth hinge.*

Usually such bodies are supported at *two points.* When the loads are all parallel, the reactions at these points are generally parallel. In most cases the positions of the supports are given and the problem is that of finding the magnitude of the reactions.

The resultant of all the parallel loads may be found by the methods of Art. 76. This resultant is equal and opposite to the equilibrant, which, in turn, is the resultant of the reactions. The problem of finding these reactions resolves itself into the problem of finding two parallel forces which have a given resultant and which pass through known points. It is the converse of the problem of finding the resultant of two known forces.

Figure 124 gives the method of finding the resultant of two reactions, $\mathbf{R}_1$ and $\mathbf{R}_2$. Suppose that the *magnitude* and *position* of the resultant **S** are known; that the *positions* of the reactions $\mathbf{R}_1$ and $\mathbf{R}_2$ are also known; and that it is desired to find the *magnitude* of these reactions. First, draw the space diagram, giving the positions of **S**, $\mathbf{R}_1$, and $\mathbf{R}_2$. Then, begin the force polygon by laying off the line *ca* of length and direction equal to the resultant. Choose a point as the pole and draw the rays *ao* and *co*. Next, choose a point No. 1 on the line of action of $\mathbf{R}_1$ and draw a line parallel to the ray *ao* intersecting the line of the resultant at point No. 3. Through this point draw a line parallel to the ray *co* and locate its point of intersection with the line of $\mathbf{R}_2$. This is point No. 2. Finally, join point No. 1 with point No. 2, and draw a line through the pole parallel to the line

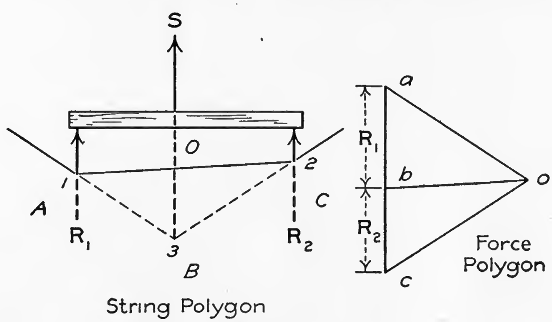


FIG. 124.

1-2. This line intersects the line *ac* at the point *b*. The length *ab* on the force polygon gives the left reaction, and the length *bc* gives the right reaction.

Problems

1. A horizontal beam 12 feet long, weighing 60 pounds, with its center of mass 5 feet from the left end, is supported at the ends. Find the reactions of the supports graphically. Check by moments.

2. A beam 20 feet long, weighing 50 pounds, has its center of mass 8 feet from the left end. It is supported by means of vertical ropes at the ends. Find the tension in each rope.

3. A horizontal beam 12 feet long, weighing 40 pounds, with its center of mass at the middle, is supported by vertical ropes at the ends. It carries a load of 30 pounds 3 feet from the left end, and a load of 50 pounds 2 feet from the right end. Find the reactions.

The resultant force may be found by the methods of Art. 76; and the reactions, as in the Example above. The two processes are generally combined. Figure 125 shows the method. On the space diagram draw the lines of the loads and the reactions. Then lay off the loads on the force polygon beginning with the load at the left of the space diagram. Choose a pole and draw the rays. Beginning with the left load of 30 pounds select point No. 1. Draw a line through point No. 1 parallel to *bo*. This line intersects the line of the 40-pound load at point No. 2. Through point No. 2 draw the next line parallel to the ray *co*, intersecting the line of the 50 pound load at point No. 3. Through point No. 1 draw a line parallel to *ao* and through point No. 3 draw a line parallel to *do*. If these lines were carried downward, their intersection would give a point on the line of the resultant. All this is exactly the same as the method of Art. 76.

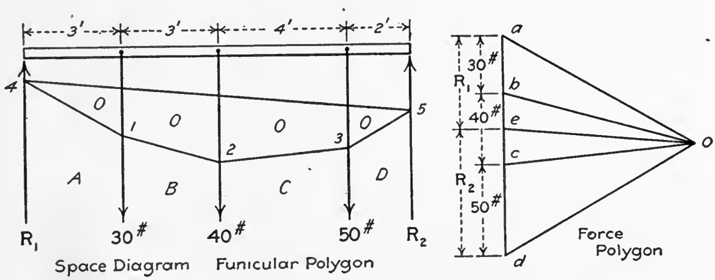


Fig. 125.

It is not necessary, however, to find the location of the resultant. The line *ad* of theforce polygon gives its magnitude and direction. The rays *ao* and *do* are components of the resultant. Draw a line through point No. 1 parallel to the ray *ao* and extend it till it intersects the line of the left reaction at point No. 4. Similarly, draw a line through point No. 3 parallel to the ray *do* and extend it till it intersects the line of the right reaction at point No. 5. Since these lines would intersect on the resultant if they were extended in the opposite directions, and since their directions are those of the first and last rays, *ao* and *do*, they correspond to the lines of Fig. 124. Connect point No. 4 to point No. 5 by a straight line and draw a ray parallel to this line. The intersection of this ray with the load line divides *ad* into two parts *ae* and *ed*. The lengths of these parts give the reactions.

Problems

4. A beam 20 feet long, with its center of mass at the middle, weighs 60 pounds. It is supported in a horizontal position by a vertical rope at each end, and carries a load of 40 pounds 4 feet from the left end, and a load of 50 pounds 4 feet from the right end. Find the reactions graphically and check by moments.

5. Solve Problem 1 if the second rope is 6 feet from the right end.

6. Solve Problem 3 if the beam is supported at the left end and 4 feet from the left end.

7. The beam of Problem 4 has one support at the left end, where the vertical reaction is 80 pounds. Locate the second support graphically. Check by moments.

79. Non-parallel Reactions. When the loads applied to a rigid body are not parallel, the reactions usually are not parallel. Sometimes when the loads *are* parallel the reactions are not parallel. Generally, the line of action of one reaction, and the location of a single point in the line of the other are known. The method of solution is the same as for parallel reactions, except that *the point of beginning of the funicular polygon is fixed by the conditions of the problem.*

Example

A beam 10 feet long, weighing 15 pounds, with its center of mass at the middle, is hinged at the left end and supported at the right end by a cord which makes an angle of 30 degrees to the left of the vertical. The right end of the beam is elevated 10 degrees above the horizontal. The beam

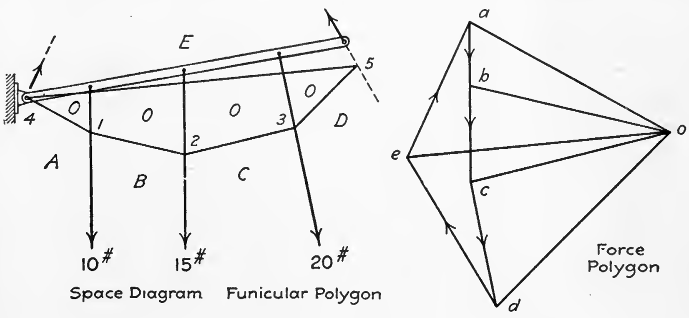


FIG. 126.

carries a vertical load of 10 pounds 2 feet from the left end, and a load of 20 pounds at an angle of 15 degrees to the right of the vertical downward at a point 2 feet from the right end. Solve for the reactions graphically, using the scale of 1 inch = 2 feet on the space diagram, and 1 inch = 10 pounds on the force diagram.

First, construct the load line and the rays as in the preceding problems. In Fig. 125, the line of action of the left reaction was known and the point No. 4 could fall anywhere on that line. In Fig. 126, the left reaction passes through the hinge, but its direction is not known. It is necessary, therefore, to put point

No. 4 at the hinge. Draw the lines 4–1, 1–2, 2–3, and 3–5 parallel to the rays *ao, bo, co* and *do* as in Fig. 125. Through *d,* on the force diagram, draw a line parallel to the direction of the right reaction. Draw the closing line 4–5 and draw a ray parallel to it. This ray intersects the line from *d* at the point *e.* The length *de* gives the magnitude of the right reaction. Draw the closing line *ea* of the force polygon. This line gives the direction and magnitude of the hinge reaction. The broken line through the hinge may now be drawn parallel to *ea.*

Problems

1. A horizontal beam 12 feet long, weighing 20 pounds, with its center of mass at the middle, is hinged at the right end and supported by a cord at the left end. The cord makes an angle of 20 degrees to the left of the vertical. The beam carries a load of 16 pounds downward 2 feet from the left end, and a load of 18 pounds 20 degrees to the left of the vertical downward 3 feet from the right end. Find the pull on the cord and the direction and magnitude of the hinge reaction. Check by moments and resolutions.

2. Find the reactions at the supports of the truss of Fig. 98.

3. A rectangular frame in a vertical plane is 10 feet wide horizontally and 4 feet high. It is hinged at the upper left corner and supported by a rope at the upper right corner. The rope makes an angle of 20 degrees to the left of the vertical. The frame weighs 160 pounds and its center of mass is at the middle. A force of 100 pounds at an angle of 10 degrees to the left of the vertical downward is applied at the lower left corner. A force of 120 pounds at an angle of 15 degrees to the left of the vertical upward is applied at the upper edge at a distance of 4 feet from the upper right corner. A force of 150 pounds downward is applied at the lower edge at a distance of 2 feet from the lower right corner. Find the tension in the rope and the direction and magnitude of the hinge reaction. Check by moments and resolutions.

CHAPTER VIII

FLEXIBLE CORDS

80. The Catenary.—A flexible cord or chain suspended from two points takes a definite form, depending upon its length and the relative positions of the points of support. If the chain or cord is of uniform weight per unit of length, the curve in a vertical plane is a *catenary*. The problem of finding the equation of the catenary is an important application of statics.

Figure 127 shows a flexible cord suspended from two points A and B. In this figure the points are at the same level. This, however, is not necessary. The cord weighs w pounds per unit

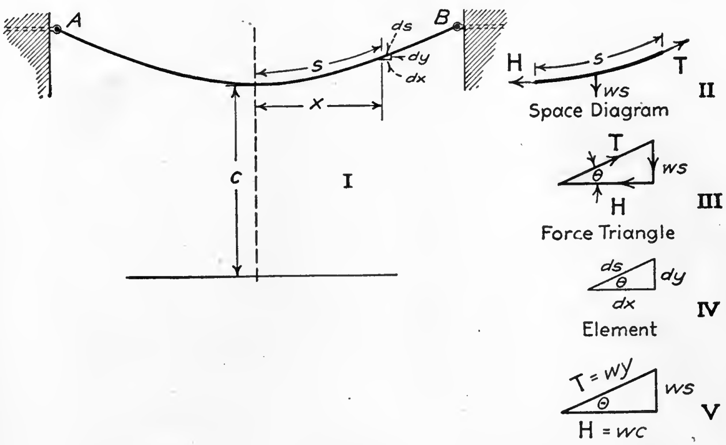


FIG. 127.

of length. The force in the cord at every point is a tension parallel to its length. At the lowest point this tension is horizontal. The horizontal component of the tension is the same at all points. In the formulas which follow, this horizontal component is represented by H. It is customary to represent $\frac{H}{w}$ by a single letter c.

$$\frac{H}{w} = c; \; H = wc$$

The constant c is the length of cord whose weight is equal to the horizontal tension (Fig. 128).

In Fig. 127, a portion of cord of length s is taken as the free body. The left end of the portion is at the lowest point of the curve, where the tension is horizontal. Figure 127, II, shows this portion of the cord detached from the remainder. Three forces act on the portion. The horizontal tension **H** is toward the left at the lower end. The tension **T** in the direction of the tangent acts at the upper right end. The weight of the portion of cord is ws and acts vertically downward. Figure 127, III, gives the force triangle for these forces. In Figure 127, I, an element of length ds is shown at the right end of the length s. The components of the element are dx and dy. Figure 127, IV, shows the same element enlarged.

Since the tension at the end of the cord is along the tangent at that position, **T** is parallel to ds and the force triangle of Fig. 127, III, is similar to Fig. 127, IV. These triangles furnish relations necessary for finding the equations of the catenary.

81. Deflections in Terms of the Length.—From the triangles of Fig. 127,

$$\frac{dy}{ds} = \frac{ws}{T}, \tag{1}$$

This equation involves three variables, T, s, and y. The tension may be eliminated by substituting

$$T = \sqrt{w^2s^2 + H^2}, \tag{2}$$

Equation (1) becomes

$$\frac{dy}{ds} = \frac{ws}{\sqrt{w^2s^2 + w^2c^2}} = \frac{s}{\sqrt{s^2 + c^2}}. \tag{3}$$

Separating the variables in Equation (3),

$$dy = \frac{sds}{\sqrt{s^2 + c^2}}, \tag{4}$$

Integrating,

$$y = \sqrt{s^2 + c^2} + K_1, \tag{5}$$

in which K_1 is an arbitrary constant of integration. If the origin is taken at the lowest point of the curve, so that $y = 0$ when $s = 0$, the value of K_1 is found to be equal to $-c$. It is customary to take the origin at a distance c below the lowest point of the curve, so that $y = c$ when $s = 0$. When these values are substituted in Equation (5), then K_1 is found to be equal to 0, and Equation (5) reduces to

$$y = \sqrt{s^2 + c^2}, \tag{6}$$

$$y^2 = s^2 + c^2 \qquad \text{Formula VII}$$

A point on the curve at a distance s from the lowest point, measured along the curve, is at a vertical distance $y - c$ above the lowest point of the curve.

A right triangle may be drawn for which y is the hypotenuse ~~and~~ s and c are the sides. In Fig. 127, III, the altitude is ws and the base is H, which is equal to wc.

$$T^2 = w^2s^2 + w^2c^2 = w^2(s^2 + c^2) = w^2y^2, \tag{7}$$

$$T = wy. \tag{8}$$

These relations are shown in Fig. 127, V. From this figure, it may be shown that

$$y = c \sec \theta, \tag{9}$$

$$s = y \sin \theta. \tag{10}$$

Example

A rope 100 feet long is stretched between points at the same level. The rope weighs 0.2 pound per foot and the horizontal component of the tension is 50 pounds. Find the sag at the middle.

Since the supports are at the same level, the lowest point is at the middle. To find the elevation of the ends above the middle, the value of s is taken as one-half the length.

$$s = 50 \text{ ft.},$$

$$c = \frac{50}{0.2} = 250 \text{ ft.},$$

$$y^2 = 50^2 + 250^2,$$

$$y = 50\sqrt{26} = 254.95 \text{ ft.}$$

$$sag = y - c = 254.95 - 250 = 4.95 \text{ ft.}$$

Problems

1. A chain weighing 240 pounds is 120 feet long, and is stretched between points at the same level. The horizontal tension is 400 pounds. Find the sag and the resultant tension at the ends.

Ans. Sag = 8.806 ft., T = 417.6 lb.

2. In Problem 1, find the sag below the support of a point on the chain which is 20 feet from one support. *Ans.* 4.845 ft.

3. A chain 100 feet in length, weighing 400 pounds, is stretched between points at the same level. The resultant tension is 520 pounds at either end. Find the horizontal tension and the sag. *Ans.* Sag = 10 ft.

4. A rope 100 feet in length is stretched between points at the same level and sags 10 feet at the middle. Find the length of rope whose weight is equal to the horizontal tension, and find the slope of the rope at the ends.

From Formula VII,

$$y^2 - c^2 = 2500;$$
$$y - c = 10,$$
$$y + c = 250,$$
$$2c = 240, \quad c = 120 \text{ ft.}$$
$$\text{slope} = \frac{50}{120} = 0.4167.$$

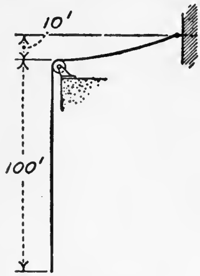


FIG. 128.

5. A flexible rope has one end attached to a fixed point. The rope runs over a smooth pulley. At the upper point of contact wth the pulley, the rope is horizontal and is 10 feet below the fixed point. The free end of the rope hangs vertically downward a distance of 100 feet (Fig. 128). Find total length. *Ans.* Total length = 145.8 ft.

6. Show that $\frac{T}{H} = \frac{y}{c}$.

82. Deflection in Terms of the Span.—From the triangles of Fig. 127,

$$\frac{dy}{dx} = \frac{ws}{H} = \frac{s}{c}. \tag{1}$$

Substituting $s = \sqrt{y^2 - c^2}$ to reduce the variables to two,

$$\frac{dy}{dx} = \frac{\sqrt{y^2 - c^2}}{c}, \tag{2}$$

$$\frac{dy}{\sqrt{y^2 - c^2}} = \frac{dx}{c}. \tag{3}$$

Integrating Equation (3),

$$\log\left(y + \sqrt{y^2 - c^2}\right) = \frac{x}{c} + K_2. \tag{4}$$

It has already been assumed that $y = c$ at the lowest point. If the Y axis is taken through the lowest point, then $x = 0$ when $y = c$. Substituting in Equation (4),

$$\log c = K_2,$$

and the equation becomes,

$$\log \frac{y + \sqrt{y^2 - c^2}}{c} = \frac{x}{c} = \log \frac{y + s}{c}; \tag{5}$$

$$\frac{y + \sqrt{y^2 - c^2}}{c} = e^{\frac{x}{c}}; \tag{6}$$

$$y + \sqrt{y^2 - c^2} = ce^{\frac{x}{c}}; \tag{7}$$

$$y^2 - c^2 = c^2e^{\frac{2x}{c}} - 2cye^{\frac{x}{c}} + y^2; \qquad (8)$$

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right). \qquad \text{Formula VIII}$$

Formula VIII is the principal equation of the catenary.
Differentiating Formula VIII,

$$\frac{dy}{dx} = \frac{e^{\frac{x}{c}} - e^{-\frac{x}{c}}}{2}. \qquad (9)$$

From Equations (1) and (9),

$$s = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right). \qquad \text{Formula IX}$$

The student who is familiar with hyperbolic functions will recognize $\frac{e^{\frac{x}{c}} + e^{-\frac{x}{c}}}{2}$ as the hyperbolic cosine of $\frac{x}{c}$ and $\frac{e^{\frac{x}{c}} - e^{-\frac{x}{c}}}{2}$ as the hyperbolic sine of $\frac{x}{c}$. Formula VIII may be written, $y = c \cosh \frac{x}{c}$; Formula IX may be written $s = c \sinh \frac{x}{c}$. Where tables of these functions are available, the labor of solving some of the following problems is materially reduced.

Example

A rope weighing 0.4 pounds per foot is stretched between points 100 feet apart at the same level. The horizontal tension is 120 pounds. Find the sag and the length of the rope.

This is a case where x and c are given. For the entire rope, $x = 50$ and $c = 300$.

$$y = 150(e^{\frac{1}{6}} + e^{-\frac{1}{6}});$$

$$\log_{10} e^{\frac{1}{6}} = \frac{0.4342945}{6} = 0.0723824;$$

$$e^{\frac{1}{6}} = 1.1813.$$

$$\log_{10} e^{-\frac{1}{6}} = \bar{1}.9276176;$$

$$e^{-\frac{1}{6}} = 0.8465$$

$$y = 150 \times 2.0278 = 304.17 \text{ ft.}$$

$$y - c = 4.17 \text{ ft.}$$

If a table of hyperbolic functions were available, the solution would be $y = 150 \cosh 0.16667$.

To find the length of half the rope, Formula VII might now be used. Since the values of $e^{\frac{1}{6}}$ and $e^{-\frac{1}{6}}$ have already been found, Formula IX is more convenient.

$$s = 150(1.1813 - 0.8465) = 50.02 \text{ ft.}$$

Problems

1. A cable weighing 1 pound per foot is stretched between points 400 feet apart at the same level. The horizontal tension is 800 pounds. Find the sag at the middle, the length of the cable, and the direction and magnitude of the resultant tension at each end.

2. A chain 100 feet in length is stretched between points at the same level and sags 5 feet. Find the horizontal distance between the points.

Solve first for c and y, as in Problem 4 of the preceding article. $c = 247.5$ ft., $y = 252.5$ ft.

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right);$$

$$s = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right);$$

$$y + s = c\,e^{\frac{x}{c}}$$

which is the same as equation (7).

$$e^{\frac{x}{c}} = \frac{302.5}{247.5};\ \log_e \frac{302.5}{247.5} = \frac{x}{c};$$

$$\log_{10}\frac{302.5}{247.5} = 0.08715,\ \log_e \frac{302.5}{247.5} = 0.200669.$$

$$2x = 0.200669 \times 2 \times 247.5 = 99.33 \text{ ft.}$$

The second form of Equation (5) gives the same method.

3. A steel tape 100 feet long is stretched between points at the same level and sags 2 feet. What is the horizontal distance between the points? If the tape weighs 1 pound, what is the horizontal tension?

Ans. $2x = 99.89$ ft.; $H = 6.24$ lb.

4. Solve Problem 3 if the sag is only 1 foot.

5. A steel tape 100 feet long, weighing 1 pound, is stretched between points at the same level. The horizontal tension is 12 pounds. Find the distance between the points.

In this problem, s is given and x is to be found. Formula VII might be used to find y, and this value of y might be substituted in Equation (6). Another method is by Formula IX.

$$1 = 12(e^{\frac{x}{c}} - e^{-\frac{x}{c}}).$$

Multiplying by $e^{\frac{x}{c}}$

$$e^{\frac{x}{c}} = 12e^{\frac{2x}{c}} - 12,$$

$$12e^{\frac{2x}{c}} - e^{\frac{x}{c}} - 12 = 0.$$

Solving this quadratic,

$$e^{\frac{x}{c}} = 1.042534,$$

from which

$$2x = 99.96 \text{ ft.}$$

6. A chain 120 feet long, weighing 100 pounds, is stretched between points at the same level. The resultant tension at either end is 130 pounds. Find the horizontal distance between the points and the sag at the middle.

83. Solution by Infinite Series.—In the problems of the preceding articles, the values of c were easily calculated from the tension and weight per foot or from the length and the sag. Problems for which c can not be so calculated are more difficult to solve.

Example

A rope is stretched between points at the same level 100 feet apart and sags 4 feet. Find the tension in terms of the weight of 1 foot of rope.

$$y = \frac{c}{2}\left(e^{\frac{50}{c}} + e^{-\frac{50}{c}}\right); \tag{1}$$

$$y - c = 4 = \frac{c}{2}\left(e^{\frac{50}{c}} + e^{-\frac{50}{c}}\right) - c, \tag{2}$$

$$8 = c\left(e^{\frac{50}{c}} + e^{-\frac{50}{c}} - 2\right). \tag{3}$$

Equation (3) may be solved by the method of trial and error. The labor is much reduced if a suitable table of hyperbolic cosines is available. An approximate value of c may easily be obtained. Where the sag is small, s is very nearly equal to x. Taking $s = 50$, and $y - c = 4$ the value of c is found to be 310.5 ft. Substituting in Equation (3),

$$8 = 310.5(1.17472 + 0.85126 - 2) = 8.067.$$

A closer approximation may be obtained by assuming that $s^2 = 2500 + 16$. This assumption gives $c = 312.5$ ft. When this value of c is substituted in Equation (3), the result is

$$8 = 8.015.$$

Exterpolating, $c = 313.1$ ft.

To get an algebraic expression for s in terms of x and c, expand

$$s = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)$$

into a series by means of Maclaurin's Theorem. This theorem is

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{\underline{2|}} + f'''(0)\frac{x^3}{\underline{3|}} + ,$$

in which $f(x)$ is the quantity required, $f(0)$ of is the value $f(x)$

when x becomes 0, $f'(0)$is the value of the first derivative of $f(x)$ when x becomes 0, etc.

Expand $e^x - e^{-x}$ by Maclaurin's Theorem.

$$f(x) = e^x - e^{-x}, \; f(0) = 0;$$
$$f'(x) = e^x + e^{-x}, \; f'(0) = 2;$$
$$f''(x) = e^x - e^{-x}, \; f''(0) = 0;$$
$$f'''(x) = e^x + e^{-x}, \; f'''(0) = 2.$$

$$e^x - e^{-x} = 0 + 2x + 0 + \frac{2x^3}{\underline{3|}} + 0 + \frac{2x^5}{\underline{5|}} + \text{etc.}$$

$$\frac{e^x - e^{-x}}{2} = \sinh x = x + \frac{x^3}{\underline{3|}} + \frac{x^5}{\underline{5|}} + \frac{x^7}{\underline{7|}} + \text{etc.} \qquad (4)$$

Equation (4) is the same as the series for sin x, except that all the terms are positive. In the series for sin x, the terms are alternately positive and negative.

Substituting $\frac{x}{c}$ for x and multiplying by c,

$$s = c\frac{e^{\frac{x}{c}} - e^{-\frac{x}{c}}}{2} = c\left(\frac{x}{c} + \frac{x^3}{\underline{3|}\,c^3} + \frac{x^5}{\underline{5|}\,c^5} + \text{etc.}\right), \qquad (5)$$

$$s = x\left(1 + \frac{x^2}{\underline{3|}\,c^2} + \frac{x^4}{\underline{5|}\,c^4} + \text{etc.}\right). \qquad (6)$$

In a similar manner,

$$y = c\frac{e^{\frac{x}{c}} + e^{-\frac{x}{c}}}{2} = c\left(1 + \frac{x^2}{\underline{2|}\,c^2} + \frac{x^4}{\underline{4|}\,c^4} + \text{etc.}\right); \qquad (7)$$

$$y - c = c\left(\frac{x^2}{\underline{2|}\,c^2} + \frac{x^4}{\underline{4|}\,c^4} + \frac{x^6}{\underline{6|}\,c^6} + \text{etc.}\right); \qquad (8)$$

$$y - c = \frac{x^2}{2c} + \frac{x^4}{24c^3} + \text{etc.} \qquad (9)$$

When x is small compared with c, as is the condition in cables with small sag, these series converge rapidly and two terms are generally sufficient to obtain a fairly accurate result.

Problems

1. A rope weighing 0.2 pounds per foot is stretched between points at the same level which are 160 feet apart. The horizontal tension is 80 pounds. Find the sag at the middle by one term of Equation (8). Solve also by means of two terms of Equation (8). Compare the results with those obtained by Formula VIII. *Ans.* 8 ft.; 8.026 ft.; 8.026 ft.

2. Find the length of the rope of Problem 1 by the series and by Formula IX. *Ans.* 161.07 ft.; 161.11 ft.

3. A chain 102 feet long is stretched between points 100 feet apart at the same level. Find the horizontal tension in terms of the weight of unit length of chain.

In this problem s and x are given and c is unknown. A problem of this kind can not be solved directly by the exponential equations of the catenary. If two terms of the series of Equation (6) are used,

$$1.02 = 1 + \frac{50^2}{6c^2},$$

from which $c = 144.3$ ft. If three terms of the series are used,

$$0.02 = \frac{50^2}{6c^2} + \frac{50^4}{120c^4},$$

from which $c = 144.7$ ft.

4. Solve the example at the beginning of this article by means of one term of Equation (9).

5. A chain weighing 1 pound per foot is stretched between points 120 feet apart at the same level. Find the sag at the middle when the horizontal tension is 180 pounds. Solve by one term of the series and by Formula VIII.

6. Solve Problem 5 if the horizontal tension is 60 pounds.

84. Cable Uniformly Loaded per Unit of Horizontal Distance. A suspension bridge consists of trusses supported by cables.

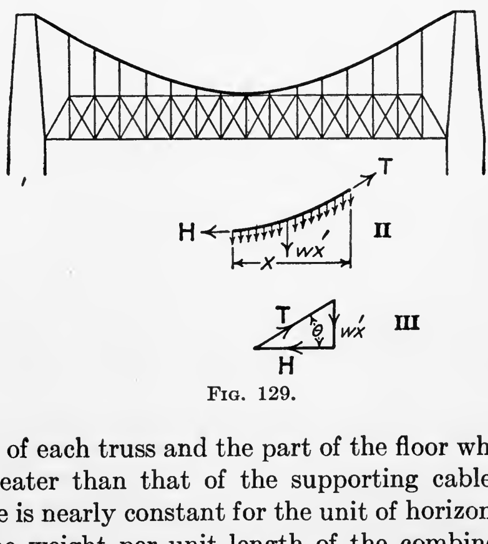


FIG. 129.

The weight of each truss and the part of the floor which it carries is much greater than that of the supporting cable. The load on the cable is nearly constant for the unit of horizontal distance. If w' is the weight per unit length of the combined truss and cable, the total weight in length x is $w'x$. Figure 129, II, shows a part of such a cable. The distance x is measured from the lowest point. The forces acting on the portion of cable are the horizontal tension **H**, the tension **T** in the direction of the tangent at the other end of the portion, and the vertical load $w'x$. Figure 129, III, is the force triangle.

$$\text{Tan } \theta = \frac{dy}{dx} = \frac{w'x}{H} = \frac{x}{c'}, \tag{1}$$

if $\frac{H}{w'} = c'$

$$dy = \frac{xdx}{c'}; \tag{2}$$

$$y = \frac{x^2}{2c'} + K.$$

If the origin of coördinates is taken at the lowest point, $y = 0$ when $x = 0$, and $K = 0$.

$$y = \frac{x^2}{2c'}. \tag{3}$$

This is the equation of a parabola with the axis vertical.

To find the length of this parabola,

$$ds = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} dx = \left(1 + \left(\frac{x}{c}\right)^2\right)^{\frac{1}{2}} dx. \tag{4}$$

Expanding Equation (4) by the binomial theorem,

$$ds = \left(1 + \frac{x^2}{2c'^2} - \frac{x^4}{8c'^4} + \frac{x^6}{16c'^6} - \frac{5x^8}{128c'^8}\right) dx \tag{5}$$

Integrating Equation (5)

$$s = x + \frac{x^3}{6c'^2} - \frac{x^5}{40c'^4} + \frac{x^7}{112c'^6} - \frac{5x^9}{1152c'^8} + \text{etc.} \tag{6}$$

A comparison of Equation (6) for the parabola with Equation (6) for the catenary shows that the first two terms of the series are identical if $c = c'$. Comparison of Equation (3) for the parabola with Equation (9) for the catenary shows that the parabola is equivalent to the first term of the series for the catenary. There are few problems to which the parabola formula strictly applies. Engineers use it frequently as an approximation for the catenary. When so used, it is better to consider it as the first term of the series of Equation (9) of Art. 83, rather than to assume that the load is uniformly distributed per unit of horizontal distance.

Do not make the common mistake of using an approximate formula for a problem, when it is easy to solve by the correct catenary equations.

Problems

1. A steel cable weighing 1 pound per foot carries a lead-sheathed telephone cable weighing 3 pounds per foot. The lead cable is suspended from the steel cable in such a way that it is practically horizontal. (This is not generally done with telephone cables.) Assuming that the combined weight per horizontal foot is constant, find the tension in the steel cable if the span is 400 feet and the sag is 20 feet.

Ans. $c' = 1000$ ft.; $T = 4080$ lb.

2. What must be the height of the towers to carry cables such as those of Problem 1 across a river 800 feet wide if the maximum tension is 5000 pounds and if the cable must be at least 40 feet above the water?

85. Summary.—In the case of a flexible cord or chain with uniform load per unit of length, the following cases are important:

(1) *Span and Horizontal Tension Given.*—Substitute in Formula VIII for the sag and in Formula IX for the length. Check by Formula VII.

For an approximate value of the sag use Equation (9) of Art. 83. The first term of the series is equivalent to the equation for a cord with uniform load per unit of horizontal distance, and gives a fair approximation when the sag is small compared with the length. For more accurate results use two or three terms. For an approximate value of the length, use Equation (6) of Art. 83.

(2) *Length and Sag Given.*—Solve for c by Formula VII, and then for x by Equation (5) of Art. 82, or by Formula VIII.

(3) *Length and Horizontal Tension Given.*—Substitute the values of s and c in Formula IX. Multiply by $e^{\frac{x}{c}}$ to form a quadratic in which the unknown is $e^{\frac{x}{c}}$. Solve for y by Formula VII.

The term $e^{-\frac{x}{c}}$ may be eliminated by adding Formulas VIII and IX instead of by multiplying by $e^{\frac{x}{c}}$.

(4) *Length and Total Tension Given.*—Solve for the horizontal tension by means of the force triangle; then use the methods of Case (3) above.

(5) *Length and Span Given.*—In this case c is unknown and Formula VII does not apply. Use the series of Art. 83 as in Problem 3 of that article.

(6) *Sag and Span Given.*—Use Equation (9) of Art. 83 to get the value of c. The first term of the series is the same as the equation of a cord with uniform load per unit horizontal length. The use of the next term involves the solution of a cubic.

After c has been found, the length may be computed by Formula VII or by the series for s.

86. Miscellaneous Problems

1. A flexible cable, weighing 1 pound per foot, is stretched across a river 1200 feet wide. The allowable tension is 6000 pounds. What must be the height of the supports if the cable is not permitted to come within 60 feet of the water? Solve as a catenary.

2. A flexible cord 200 feet long, is stretched between two points of support. The lowest point of the cord is 8 feet below one of the points and 12 feet below the other. Find the horizontal distance between the supports.

Ans. Lowest point is 89.8 ft. from the lower support, measured along the cord. The total horizontal distance between supports = 198.66 ft.

3. A cable across a river 800 feet in width is used to supply concrete for a dam. The cable weighs 2 pounds per foot. The allowable tension is 10,000 pounds. The maximum load is 1200 pounds. Find the sag at the middle when this load is at the middle. Solve by approximate formula. Consider the additional load as replaced by additional cable in the middle.

Ans. Sag at middle = 40.4 ft. with no allowance for stretch of the cable.

CHAPTER IX

CONCURRENT NON-COPLANAR FORCES

87. Resultant and Components.—It is an experimental fact that the resultant of two concurrent, coplanar forces is represented by their vector sum. Starting from this fact, it has been shown that the resultant of any number of concurrent, coplanar forces may be represented by their vector sum. Finally, it has been proved that the direction and magnitude of the resultant of any number of non-concurrent, coplanar forces may be determined from the vector sum of these forces.

The next problem for consideration is that of *concurrent, non-coplanar* forces. Figure 130 shows two forces, **P** and **Q** in a vertical plane, and a single force **H** in the horizontal plane. The resultant of **P** and **Q**, or of any number of forces in the same plane, is the single force **R** in that plane. This resultant **R** and the force **H** intersect at the point of application. They lie in the inclined plane *OAB* of Fig. 130, II. Since **R** and **H** are in a plane, their resultant is their vector sum in that plane. The force **H** as well as the force **R** may be the resultant of several forces in its plane. In Fig. 130 the original planes are normal to each other. There is nothing in the proof which is limited to this position of the planes. They may have any angle with each other.

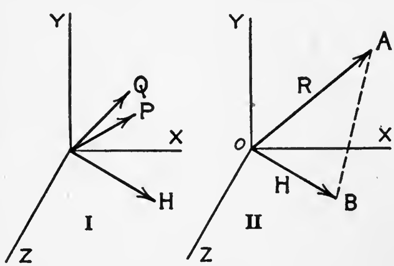


Fig. 130.

The resultant of concurrent forces in space may be represented by their vector sum. The forces which make up the resultant are its *components.*

88. Resolution of Non-coplanar Forces.—The most important components are those obtained by orthographic projection upon a line or upon a plane. If the force is projected upon three lines at right angles to each other, the three components form the edges of a rectangular parallelopiped. The resultant of these three components is the diagonal of the parallelopiped.

Example

A cord is attached to a fixed point A, passes over a pulley B, and carries a load of 24 pounds on the free end. The pulley is 4 feet south and 3 feet east of a point C. The point C is 6 feet above the point A. Find the components of the pull in the cord vertically upward, horizontally toward the south, and horizontally toward the east,

The length of the diagonal from A to B is $\sqrt{61}$ ft. East component $= \frac{3}{\sqrt{61}} \times 24 = 9.22$ lb. South component $= \frac{4}{\sqrt{61}} \times 24 = 12.29$ lb. Vertical component $= 18.44$ lb.

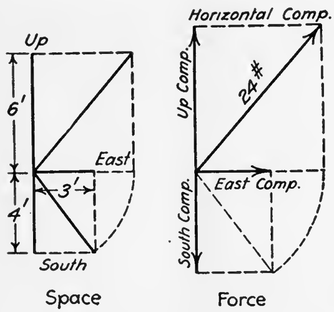


FIG. 131.

It will be observed that the components are proportional to the edges of a rectangular parallelopiped of which the dimensions are 3 ft., 4 ft., and 6 ft.

Figure 131 gives the graphical solution for these components.

The student should make a sketch of the cord and pulley.

Problems

1. A horizontal trap-door, 6 feet by 10 feet is lifted by a rope attached to one corner. The rope passes over a pulley, which is 7 feet above the middle of the door, and carries a load of 40 pounds on the free end. Find the vertical component of the pull on the door and find the horizontal components parallel to the edges. *Ans.* Vertical component = 30.73 lb.

2. Solve Problem 1 graphically, using the scale of 1 inch = 2 feet in the space diagram, and 1 inch = 10 pounds in the force diagram.

3. A pull of 50 pounds is applied by a rope which makes an angle of 35 degrees with the vertical in a vertical plane wh˙ch is north 20 degrees east. Find the vertical component of the force and the horizontal components east and north. *Ans.* Horizontal component north = 26.95 lb.

4. Solve Problem 3 graphically to the scale of 1 inch = 10 pounds.

5. A force of 20 pounds makes an angle of 42 degrees with the vertical, and an angle of 55 degrees with the horizontal line toward the north. Find its vertical component and its horizontal components east and north.

Ans. Horizontal component east = 6.89 lb.

6. Solve Problem 5 graphically to the scale of 1 inch = 5 pounds.

89. Calculation of Resultant.—To find the resultant of a number of non-coplanar forces, it is best to resolve each force along each of three axes at right angles to each other, to add the components along each axis to get the component of the resultant along that axis, and finally to combine these components to form a single force. The *components* of the resultant are the edges of a rectangular parallelopiped of which the resultant is the diagonal. The resolutions and compositions may be made algebraically or graphically. In the graphical method it is best to resolve each force on a separate diagram.

Example

Find the resultant of the following three forces: 10 pounds at 38 degrees with the vertical in a vertical plane which is north 25 degrees east; 12 pounds east of north at an angle of 45 degrees with the vertical and an angle of 60 degrees with the north horizontal; and 13 pounds applied by means of a rope which passes over a pulley 4 feet east, 3 feet south, and 12 feet above the point of application of the forces.

Force	Components		
	Up	North	East
10	7.880	5.580	2.602
12	8.484	6.000	6.000
13	12.000	− 3.000	4.000
	28.364	8.580	12.602

If ϕ is the angle which the vertical plane of the resultant makes with the north vertical plane,

$$\tan \phi = \frac{12.602}{8.580},$$

$$\phi = 55^\circ 45'.$$

If H is the resultant of the two horizontal components,

$$H = \frac{12.602}{\sin \phi} = 15.25 \text{ lb.}$$

If θ is the angle which the resultant makes with the vertical,

$$\tan \theta = \frac{15.25}{28.36}, \quad \theta = 28^\circ 16'.$$

$$R = \frac{28.36}{\cos \theta} = 32.20 \text{ lb.}$$

The resultant is a force of 32.20 pounds at an angle of 28° 16′ with the vertical in a vertical plane which is north 55° 45′ east.

Problems

1. Find the resultant of 25 pounds at an angle of 42 degrees with the vertical in a vertical plane which is north 34 degrees east, and 20 pounds at an angle of 56 degrees with the vertical in a vertical plane which is north 20 degrees west. Solve by resolutions along east and north horizontal axes, and also solve with one horizontal axis north 70 degrees east and the other north 20 degrees west.

2. Solve Problem 1 graphically to the scale of 1 inch = 10 pounds. Make a separate diagram for each resolution, and a third diagram for the composition.

3. Find the resultant of the following forces: 40 pounds east of north at an angle of 50 degrees with the vertical and 60 degrees with the horizontal north; 50 pounds east of south, at an angle of 70 degrees with the vertical and 65 degrees with the horizontal south; 40 pounds above the horizontal at an angle of 62 degrees with the horizontal north and 40 degrees with the horizontal east.

Ans. 115.4 lb. at an angle of 57° 37′ with the vertical, in a vertical plane north 79° 34′ east.

90. Equilibrium of Concurrent, Non-coplanar Forces.—The condition for equilibrium of concurrent, non-coplanar forces is the same as for coplanar forces. The force polygon must close, or the resultant of all the forces must equal zero. Since the force polygon is not all in one plane, three components are required to specify fully some of the forces. There are three unknowns and three independent equations. The sum of the components along any axis is zero. In order that the equations may be independent, one of these axes must have a component perpendicular to the plane of the other two.

The principal difficulty in the solution of a problem of concurrent, non-coplanar forces is that of finding the required angles from the conditions of the problem.

Example

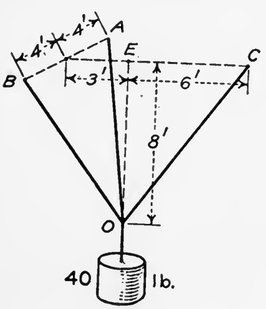


FIG. 132.

A mass of 40 pounds is supported by three ropes. The ropes are fastened to a horizontal ceiling 8 feet above the body at points A, B, and C, respectively. If E is a point in the ceiling directly over the body, the point A is 3 feet north and 4 feet east of E; the point B is 3 feet north and 4 feet west of E; the point C is 6 feet directly south of E. Find the tension in each rope.

A horizontal resolution perpendicular to the vertical plane ECO of Fig. 132 eliminates the force in OC. The ropes OA and OB make equal angles with the axis of resolution. If the forces in the ropes are represented by A, B, and C, respectively,

$$A = B.$$

By a vertical resolution,

$$2A \times \frac{8}{\sqrt{89}} + 0.8C = 40.$$

By a horizontal resolution parallel to *EC*

$$2A \times \frac{3}{\sqrt{89}} - 0.6C = 0;$$

$$A = \frac{5\sqrt{89}}{3} = 15.72 \text{ lb.}$$

$$C = 16.67 \text{ lb.}$$

Problems

1. In the example above, let *B* be 3 feet west of the plane *ECO*; let the other data remain unchanged. Find the tension in each rope.

Ans. $C = 16.67$ lb.; $A = 13.48$ lb.; $B = 17.34$ lb.

2. A 40-pound mass is supported by three ropes. Each rope is 10 feet long. The ends of the ropes are attached to a horizontal ceiling at the vertices of an equilateral triangle each side of which is 8 feet. Find the tension in each rope.

91. Moment about an Axis.—The moment of a force about an axis is the product of the component of the force in a plane perpendicular to the axis multiplied by the shortest distance from the line of the force to the axis. In Fig. 133, the force is **P** and the axis is *OZ*. The *XY* plane is perpendicular to the axis. The force **P** may be resolved into two components at right angles to each other. One component, which is not shown in Fig. 133, is parallel to the axis *OZ*. The other component is **P′** in the plane perpendicular to the axis. The distance *d* from the axis to the line of the component **P′** is the effective moment arm.

$$M = P' d.$$

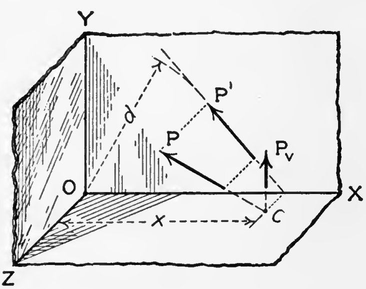


FIG. 133.

In Art. 47, the moment of a force was said to be about a point in its plane. It is better to consider the moment to be about an axis which passes through the point and which is perpendicular to the plane.

In the case of the moment of a force about a point in its plane, the apparent arm was drawn from the origin of moments to the line of the force; then this arm was multiplied by the component of the force perpendicular to it. Likewise, in the case of the moment of a force with respect to any axis, a plane may be passed through the axis of moments intersecting the line of force at any convenient point. The component of the force normal to this plane multiplied by the perpendicular distance from the point of intersection to the axis gives the required moment.

In Fig. 133, the plane OXZ passes through the axis OZ. The line of force may be extended until it intersects this plane at C. The moment of the force about OZ is the product of the vertical component $\mathbf{P}_v$ multiplied by the distance from C to the axis. The other components of the force $\mathbf{P}$ at this point lie in the plane OXZ and either intersect the axis or are parallel to it. Consequently only the vertical component $\mathbf{P}_v$ has moment about the axis.

The line of the force $\mathbf{P}$ might be extended in the opposite direction till it intersects the vertical plane OYZ. The horizontal component of the force at this point of intersection multiplied by the vertical distance from the point to the axis gives the moment. Any plane through the axis may be used in this way. The choice of a plane depends upon the difficulty of finding the component of the force perpendicular to it, and of finding the distance from the point of intersection to the axis.

Example

In Fig. 132, find the moment of the force in OC about the axis AB.

Use first the horizontal plane ABC. The moment arm is 9 feet in length. If C is the magnitude of the force in OC, its vertical component is $0.8C$.

$$M = 0.8C \times 9 = 7.2C.$$

Use next the vertical plane through the axis. Extend the line of CO beyond O till it intersects this plane. The point of intersection is 12 feet from BA and the horizontal component of the force is $0.6C$.

$$M = 0.6C \times 12 = 7.2C.$$

92. Equilibrium by Moments.—If a body is in equilibrium under the action of a number of *concurrent, non-coplanar* forces, the sum of the moments of these forces about any axis is zero. If a plane is passed through the axis of moments and the point of application of the forces, the moment of each force will be the product of its component perpendicular to the plane multiplied by the distance of the point of application from the axis. Since the moment arm is the same for all the concurrent forces, the sum of the moments is equivalent to a resolution perpendicular to the plane. It is not necessary to use this plane in calculating the moments. Any convenient plane may be used, and different planes may be employed for the different concurrent forces.

Instead of three resolution equations of equilibrium, there may be one moment and two resolutions, two moments and one resolution, or three moments.

Example

Find the force C of Fig. 132 by moments about AB. The line AB is taken as the axis because it eliminates two unknowns. The moment of C about AB has already been found to be 7.2 C. The moment of the weight of 40 pounds is $40 \times 3 = 120$ ft.-lb.

$$7.2C = 120,$$
$$C = 16.67 \text{ lb.}$$

With C known, A and B may best be determined by two resolution equations. The method of moments, however, may be employed. To find the force B take moments about the horizontal line AC. Figure 134 shows the top view, or plan of Fig. 132. The moment arm of the 40-pound mass is the line EF whose length is 6 sin θ. The moment arm of the vertical component of the force B is 8 cos θ. From the figure, tan $\theta = \frac{4}{9}$.

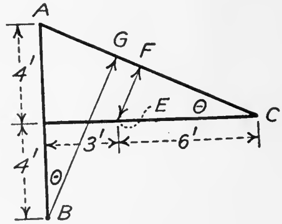


FIG. 134.

$$\frac{8B}{\sqrt{89}} 8 \cos \theta = 40 \times 6 \sin \theta;$$

$$B = \frac{15\sqrt{89}}{4} \tan \theta = \frac{5\sqrt{89}}{3} = 15.72 \text{ lb.}$$

Problems

1. Solve Problem 1 of Art. 90 by moments.

2. A mass of 60 pounds is suspended by two ropes, each 13 feet in length, attached to a horizontal ceiling at points 10 feet apart, and by a third rope, 15 feet in length, attached to the same ceiling at a point 12 feet from the point of attachment of each of the others. Find the tension in each rope by moments and check by a vertical resolution.

93. Summary.—The resultant of concurrent, non-coplanar forces is represented by their vector sum.

The moment of a force about an axis is the moment of its component in a plane perpendicular to the axis about the point of intersection of the axis with the plane. To calculate the moment, pass a plane through it and some point on the line of the force. The component of the force normal to the plane multiplied by the distance of the point from the axis gives the required moment.

For equilibrium the force polygon closes. There are three unknowns and three independent equations may be written. These equations may be:

(1) Three resolutions.
(2) One moment and two resolutions.
(3) Two moments and one resolution.
(4) Three moments.

A moment equation for non-coplanar concurrent forces is equivalent to a resolution perpendicular to the plane which passes through the axis of moments and the point of application of the forces.

In writing resolution equations, each direction of resolution must have a component perpendicular to the plane of the others. It is not necessary that these directions be at right angles to each other. A resolution perpendicular to the direction of a force eliminates that force.

When moment and resolution equations are written together, a resolution must not be taken perpendicular to the plane through the axis of moments and the point of application of the forces. When two resolutions and one moment equation are written, the plane of the directions of the two resolutions must not be perpendicular to the moment plane.

Two axes of moments and the point of application of the forces must not lie in the same plane. Three axes of moment must not be parallel.

CHAPTER X

PARALLEL FORCES AND CENTER OF GRAVITY

94. Resultant of Parallel Forces.—Articles 55 and 56 show that the resultant of two or more parallel forces in the same plane is equal to their algebraic sum, and that the sum of the moments of all the forces about any point in their plane, is equal to the moment of their resultant about that point. It will now be shown *that the resultant of parallel forces which are not all in the same plane is the algebraic sum of the forces, and that the sum of the moments of all the forces about any axis is equal to the moment of their resultant about that axis.*

Figure 135 shows two forces, **P** and **Q** together with their resultant **R**. The moment of **R** about any point in their plane

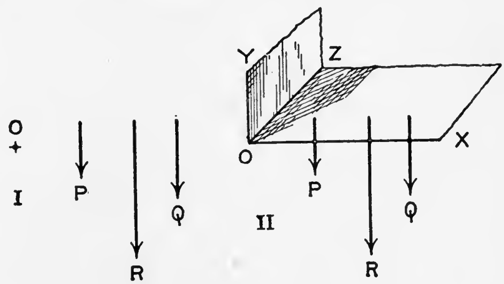


FIG. 135.

is equal to the sum of the moments of **P** and **Q** about that point. The origin of moments at the point O may be regarded as the point at which an axis perpendicular to the plane of the forces intersects that plane. In Fig. 135, II, the axis is OZ. If the moment of the resultant is equal to the sum of the moments of the two forces about an axis perpendicular to their plane, the moments of these forces about any other axis through O are likewise equal. To change from the moment about the perpendicular axis to the moment about an inclined axis, it is necessary to multiply each term by the cosine of the same angle; hence, if the moments are equal in the one case, they are equal in the other.

Let **S** be a third force which is parallel to **P** and **Q** but is not in the same plane. A plane may be passed through **R** and **S**. To find their resultant the forces, **R** and **S,** may now be treated in the same way as **P** and **Q.** *The resultant of three parallel forces which are not all in the same plane is equal to the algebraic sum of the forces and the moment of the resultant about any axis is equal to the sum of the moments of the three forces about that axis.* This method may be extended to include any number of forces. The axis of moments is generally in a plane perpendicular to the direction of the forces. This, however, is not necessary.

Example

Find the magnitude and position of the resultant of the following vertical forces: 24 pounds down through the point (3, 5), 16 pounds down through the point (2, 7), 9 pounds up through the point (5, 4), 12 pounds down through the point (−6, 3), and 23 pounds up through the point (8, −5).

In the tabulated solution below, since most of the forces are downward, that direction is taken as positive. The moments are calculated about the X and the Z axes.

FORCE	x	Px	z	Pz
24	3	72	5	120
16	2	32	7	112
− 9	5	− 45	4	− 36
12	−6	− 72	3	36
−23	8	−184	−5	115
$R = 20$		$R\bar{x} = -197$		$R\bar{z} = 347$

$$\bar{x} = -9.85,\ \bar{z} = 17.35,$$

in which $\bar{x}$ and $\bar{z}$ are the coördinates of the resultant force.

Problems

1. Find the magnitude and position of the resultant of the following horizontal forces: 184 pounds toward the east through a point 5 feet above a floor and 3 feet north of a vertical wall, 160 pounds east through a point 8 feet above the floor and 2 feet south of the wall, 124 pounds west through a point 3 feet above the floor and 6 feet north of the vertical wall.

2. A rectangular door in a horizontal position is 7 feet long from left to right and 4 feet wide. It weighs 40 pounds and its center of mass is at the center. The door carries 50 pounds at the right corner near the observer, 35 pounds at the left corner diagonally across from the 50 pounds, 24 pounds at the other right corner, and 16 pounds at the middle of the left edge. Find the location of a single support which will hold the door in the horizontal position.

95. Equilibrium of Parallel Forces.—To find the resultant of parallel forces, three quantities are calculated. These are the magnitude of the resultant and two coördinates of its position. A problem of the equilibrium of parallel forces involves three unknowns and requires three independent equations for its solution. Since the forces are all in one direction, each force is fully determined by a single resolution. Only one of the three possible equations may be a resolution equation. When there are two moment equations, one of the axes must not be parallel to the other. When three moment equations are written, not more than two of the axes may be parallel. It will be noticed that two moment equations and one resolution equation were used in the example of Art. 94. It is not necessary that the axes of moments be at right angles to each other.

Problems

1. A horizontal trap door is 8 feet long and 4 feet wide. Its center of mass is at the center. The door has 2 hinges, each one foot from a corner on an 8-foot side. The door is lifted by a vertical rope attached 3 feet from one corner on the other long side. Find the tension in the rope and the hinge reactions. Solve by three moment equations and check by a vertical resolution.

2. Solve Problem 1 if the rope is attached to one corner.

3. In the trap door of Problem 1, find the position at which the rope may be attached at the edge of the door in order that one hinge reaction may be zero. Solve algebraically and graphically.

4. A horizontal table is 5 feet long from left to right and 3 feet wide. It is supported on three legs, one at each of the left corners and one at the middle of the right side. A load of 60 pounds is placed 1 foot from the front edge and 1 foot from the left edge. Find the reaction of each leg if the weight of the table is neglected.

5. The load on the table of Problem 4 is placed 6 inches from the front edge. How far may it be moved from the left edge without upsetting the table.

6. Solve Problem 5 if the table weighs 20 pounds and its center of mass is at the center.

7. A wheelbarrow is 5 feet long from axis of axle to points at which the handles are gripped, and is 2 feet wide between centers of handles. It carries a load of 160 pounds, which is placed 1 foot from the axis of the axle and 4 inches to the right of the middle. Find the reaction of the wheel and of each handle.

96. Center of Gravity.—When parallel forces are applied to each particle of a body, and each force is proportional to the mass of the particle upon which it acts, the line of action of the resultant of these forces passes through a point which is called the *center of mass of the body.* The force of gravity on each particle of a body is directed toward the center of the Earth, which is so distant that the forces on the various particles of an ordinary body are practically parallel. The force of gravity varies as the mass of the particle and inversely as the square of its distance from the center of the Earth. This distance is so great that for bodies of ordinary dimensions, the error in the assumption that the attraction on each particle is proportional its mass is negligible. For these reasons *center of mass* is usually regarded as synonymous with *center of gravity. Center of gravity* may be defined as the point of application of the resultant of the gravity forces which act on the body. The term *center of gravity* is used more commonly than *center of mass.*

The position of a line through the center of gravity may be found by taking moments about two axes which are not parallel. The position of the center of gravity on this line is found by regarding the direction of all the forces as changed and then taking moments about a third axis.

The coördinates of the center of gravity are represented by $\bar{x}$, $\bar{y}$, and $\bar{z}$. These may be read "bar x," "bar y," and "bar z."

Example

The base ABC of Fig. 136 is a plane surface and the faces $ABDE$ and DBC are also plane and are perpendicular to each other and to the base. The body weighs 24 pounds. The body rests on three supports, which are not shown in the drawing. Two supports are 1 inch from the edge AB. The third support, 17 inches from the edge AB, rests on a platform scale. The scale reading is 4.52 pounds when the base is horizontal. How far is the center of gravity from the plane $ABDE$?

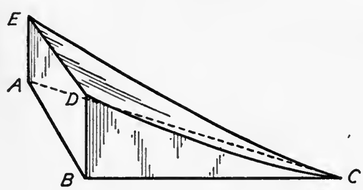


FIG. 136.

Taking moments about the two supports,

$$24\bar{x} = 4.52 \times 16,$$
$$\bar{x} = 3.01 \text{ in.}$$

The center of gravity lies in a vertical plane which is parallel to the face $ABDE$ and is 4.01 inches from it.

When the body is supported on two points, each 1 inch from the edge BC, and a third point rests on a platform scale 13 inches from BC, the scale reading is found to be 7.24 pounds. From this experiment, the center of gravity is found to lie in a plane parallel to the face BCD at a distance of 4.61 inches from that plane. The intersection of these two planes through the center of gravity gives the location of a line through the center of gravity.

To find the location of the center of gravity on the line of intersection of the two planes, its position must be changed so that the forces of gravity will act in a relatively different direction. For the experiment the body is placed with the face *BCD* horizontal. When it is supported at two points, each 1 inch from *BC*, and on a third point at a distance of 11 inches from *BC*, the scale supporting the third point reads 7.20 pounds. From this reading the center of gravity is found to be 4.00 inches from the plane *ABC*.

It will be observed that each of these experiments locates a vertical plane which is parallel to the axis of moments and passes through the center of gravity. Three such planes, if no two of them are parallel, fully locate the center of gravity.

Problems

1. A triangular board of uniform thickness is 12 inches wide at the base. The perpendicular distance from the vertex to the base is 18 inches. The board is supported on two knife-edges, each 0.5 inch from the base, and on a third knife edge, parallel to the base, at 1 inch from the vertex. The board weighs 15.00 pounds. When the third knife-edge, which weighs 0.45 pound, rests on a platform scale, the reading is 5.45 pounds. How far is the center of gravity of the board from the base? *Ans.* 6.00 inches.

2. A rectangular plate of uniform thickness is 24 inches long and 18 inches wide. A circular hole, 10 inches in diameter, is cut from the plate. The center of the hole is 6 inches from the front 24-inch edge and 8 inches from the left 18-inch edge. The plate is supported on a knife-edge 0.1 inch from the left edge and on a second knife edge at the same distance from the right edge. The knife edge at the right rests on a platform scale. The increase of the load on the scale when the plate is put on the knife-edge, amounts to 22.72 pounds. The plate weighs 42.40 pounds. How far is its center of gravity from the left edge?

As a second experiment, the plate is supported on a knife-edge 0.1 inch from the front and on a second knife-edge 17 inches from this one. The load on the second knife-edge reads 23.87 pounds. How far is the center of gravity from the front edge of the plate?

3. A heavy rectangular door is placed in a horizontal position and supported by three spherical steel balls. Each ball rests on a separate platform scale. The door is 6 feet long from left to right and 4 feet wide. One ball is 1 inch from the front edge and 1 inch from the left edge. The scale which supports it reads 32 pounds. The second ball is 1 inch from the left edge and 3 inches from the rear edge. The scale which supports it reads 104 pounds. The third ball is 1 inch from the right edge and 1 foot from the front edge. The scale which supports it reads 144 pounds. Locate the center of gravity.

Ans. One inch to the right and 1 inch to the front of the middle.

When the dimensions of a body and its density at all points are known, the center of gravity may be calculated. Moments are taken in the same way as in the experimental determination. For the last determination, the position of the body is not changed.

The direction of the parallel forces is assumed to be different. This assumption makes it possible to take moments about an axis which is perpendicular to the first two axes of moments. In Fig. 136, the forces may be regarded as parallel to the line AB.

When all the forces are parallel to a given plane, the moment about any one of a set of parallel lines in that plane is the same. In this case the moment is often called the *moment with respect to the plane.* In Fig. 136, the moment about AB is equal to the moment about any line in the plane of $ABDE$ parallel to the line AB. The moment with respect to AB is the moment with respect to the plane $ABDE$.

97. Center of Gravity Geometrically.—When the density of a body is uniform, the center of gravity may sometimes be determined geometrically. If the body has a plane of symmetry, the plane divides it into equal halves. If the plane of symmetry be placed vertical, or if the forces be regarded as acting parallel to it, the moment of one-half of the body about any axis in the plane is equal and opposite the moment of the other half. *If a body of uniform density has a plane of symmetry, the center of gravity lies in that plane.* If there are two or more planes of symmetry, the center of gravity lies at their intersection.

All planes through the center of a sphere are planes of symmetry. *The center of gravity of a sphere is at its center.*

All planes through the axis of a right circular cylinder are planes of symmetry. The plane perpendicular to the axis at the middle is also a plane of symmetry. *The center of gravity of a right circular cylinder of uniform density is located at the middle of the axis.*

The center of gravity of a rectangular parallelopiped is located at a point which is midway between all the faces.

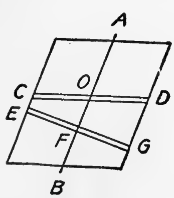


Fig. 137.

The center of gravity of a right prism, the sections of which are equilateral triangles is located at the middle of its length at the intersection of the lines which bisect the angles of the section.

Figure 137 shows an oblique-angled parallelogram. The sections perpendicular to the plane of the paper are assumed to be rectangular. The plane of which the line AB is the trace is not a plane of symmetry. It is a plane of symmetry for elements, such as EFG, perpendicular to it, but these elements do not include the entire figure. They leave out a triangular area at the top and another at the bottom. If, however, the parallelogram is divided into elements such as COD, by lines parallel to the ends, a series of such elements will include the entire figure. It will be shown that the center of gravity of the element COD is midway between the ends. If the part OC is rotated about the line AB as an axis, it will not coincide with the part OD. If, however, OC is rotated through 180 degrees about an axis at O perpendicular to the plane of the paper, it will coincide exactly with OD. The center of gravity of OC is as far from the line AB on one side as the center of gravity of OD from the line on the other side. The moment of OC about the line AB is equal and opposite to the moment of OD. The center of gravity of the element lies in the plane of AB. The center of gravity of an oblique prism or cylinder lies in a plane parallel to the bases and midway between them.

Problems

1. A plate of uniform thickness and density is of the form of a 5″ by 4″ by 1″ angle section as shown in Fig. 138. Find the distance of the center of gravity from the back of each leg. Solve by dividing the section into two equal rectangles. Check $\bar{x}$ by dividing the section into a 5″ by 1″ rectangle and a 1″ by 3″ rectangle. Since the thickness and density are constant, the terms which represent them may be omitted.

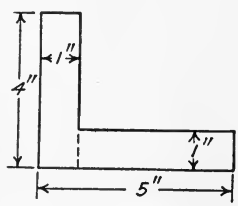


FIG. 138.

Taking moments about the back of the 4-inch leg,

AREA	ARM	MOMENT
4	0.5	2.0
4	3.0	12.0
8		14.0

$$\text{Total mass} = 8. \quad \text{Total moment} = 14$$
$$8\bar{x} = 14,$$
$$\bar{x} = 1.75 \text{ in.}$$

2. A homogeneous solid cylinder, 6 inches in diameter and 10 inches long, weighs 15 pounds. The cylinder stands on a rectangular parallelopiped, 10 inches square and 6 inches high, which weighs 35 pounds. Find the distance of the center of gravity of the two from the base of the parallelopiped.

Ans. $50\bar{y} = 270$; $\bar{y} = 5.4$ in. from the base.

3. A homogeneous solid cylinder 8 inches in diameter is 6 inches long. A second homogeneous solid cylinder is 4 inches in diameter and 10 inches long. The 8-inch cylinder stands vertical and the 4-inch cylinder is placed on it with its axis 2 inches to the right of the axis of the first cylinder. If both cylinders have the same density, find the center of gravity.

Ans. $\bar{x}$ = 0.59 in. from axis of 8-in. cylinder;
$\bar{y}$ = 5.35 in. from base of 8-in. cylinder.

4. Solve Problem 2 if the cylinder and block have the same density.

5. Solve Problem 3 if the density of the 8-inch cylinder is twice that of the 4-inch cylinder.

6. The plate of Problem 2 of Art. 96 is of uniform thickness and density. Determine the position of the center of gravity. Subtract the moment and mass of the circular hole from the moment and mass of the entire plate.

7. A homogeneous cylinder 8 inches in diameter and 10 inches long has a circular hole drilled in one end. The hole is 4 inches in diameter, 8 inches deep, and its axis is 1 inch from the axis of the cylinder. Find the center of gravity of the remainder.

98. Center of Gravity of a Triangular Plate.—Figure 139 represents a triangular plate of uniform thickness perpendicular to the

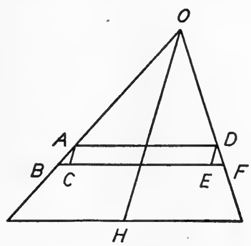


FIG. 139.

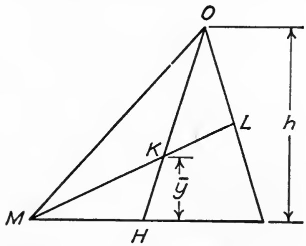


FIG. 140.

plane of the paper. *OH* is the median line. The triangle is divided into elements such as *ADFB*, by lines parallel to the base. The moment of the oblique parallelogram *ACED* with respect to the line *O H* (or any line in the plane through *OH* perpendicular to the plane of the paper) is zero. The small triangles *ABC* and *DEF* are left over. These triangles have equal areas, and their bases are equally distant from the line *OH*; consequently their moments about *OH* must be *approximately* equal. The lines *A D* and *B F* may be brought infinitely close together, so that the areas of these triangles become infinitesimals of the second order, and the difference of their moments become infinitesimals of the third order. The center of gravity of the element *ADFB* must then lie in the median line. The entire triangle is made up of similar elements, each of which has its center of gravity on the median line. It follows that the center of gravity of the entire triangular plate of uniform thickness and density must lie on the plane of the median line.

A second median may be drawn as in Fig. 140. The center of gravity lies in this median, and, consequently, at the intersection of the medians.

It was learned in Geometry that the intersection of the medians of a triangle is located at one-third the length from the base. In Fig. 140, HK is one-third of OH, and KL is one-third of ML. If h is the perpendicular distance from a vertex to the opposite base, and $\bar{y}$ is the perpendicular distance from the center of gravity to that base, then $\bar{y}$ is one-third of h.

Example

A plate of uniform thickness and density is in the form of a trapezoid The lower base is 18 inches, the upper base is 10 inches, and the height is 12 inches. Find the distance of the center of gravity from the lower base.

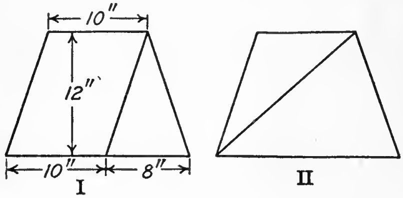


FIG. 141.

The trapezoid may be divided, as shown in Fig. 141, I, into a parallelogram and a triangle. Since the thickness and density are constant, their values in the mass and moment will cancel. Only the area, then, need be considered.

	AREA	ARM	MOMENT
Parallelogram.................	120	6	720
Triangle......................	48	4	192
	$168\bar{y}$	=	912

$$\bar{y} = 5\tfrac{3}{7} \text{ in.}$$

Problems

1. Solve the example above by dividing the trapezoid into two triangles as in Fig. 141, II.

2. Solve the example above by means of a parallelogram of 18-inch base from which an inverted triangle is subtracted to form the trapezoid.

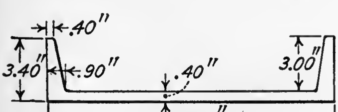


FIG. 142.

3. A trapezoidal plate of uniform thickness has the lower base 24 inches, the upper base 18 inches, and the height 15 inches. Find the distance of the center of gravity from the lower base. Check by moments about the upper base.

4. Figure 142 shows a standard 15″ channel section. Find the distance of the center of gravity from the back of the web, and compare the result with *Cambria Steel* or *Carnegie Handbook.*

5. A wooden block is 18 inches square and 10 inches high, and its specific gravity is 0.8. A metal cylinder, 5 inches in diameter and 4 inches high, stands on the block. The axis of the cylinder is 10 inches from the front face of the block and 12 inches from the left face. The specific gravity of the metal is 7.8. Locate the center of gravity of both with reference to the bottom, left, and front surfaces of the block.

Ans. $\bar{y} = 6.34$ in.; $\bar{x} = 9.57$ in.; $\bar{z} = 9.19$ in.

6. A triangular steel plate is 2 inches thick. The triangle is isosceles with a base of 18 inches and an altitude of 30 inches. Steel weighs 500 pounds per cubic foot. The plate is placed horizontally and supported by three spherical balls. One ball is 1 inch from the base and 6 inches from the bisector of the opposite angle. Another sphere is 1 inch from the base and 8 inches from the bisector. The third sphere is on the bisector at a distance of 24 inches from the base. Find the load on each sphere.

99. Center of Gravity of a Pyramid.—Figure 143 shows a triangular pyramid. A median BE is drawn in the base BCD, and a point F is located at one-third the length of the median from E. If the base is regarded as a thin triangular plate, the point F is at its center of gravity. A line AF is drawn from the vertex A to the center of gravity of the base. Let $B'C'D'$ be a thin plate between two planes which are parallel to the base BCD. This plate is similar to the base. From the similarity of the triangles it may be shown that the line AF passes through the center of gravity of this plate. Since $B'C'D'$ may be any plate parallel to the base, it follows that the center of gravity of the pyramid is located on the line AF.

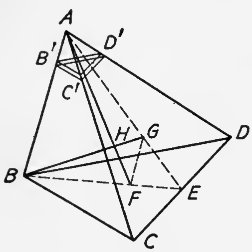


FIG. 143.

In the face ACD the median AE is drawn and the point G is located on this median at one-third of its length from E. The line BG, from the vertex opposite the face ACD, passes through the center of gravity of the pyramid. The lines AF and BG are in the plane AEB and intersect at the point H in that plane.

The triangles ABE and GFE have the same angle at E. The side GE is one-third of AE and the side FE is one-third of BE. The triangles are similar and the side GF is parallel to AB and one-third its length. In the triangles BAH and GFH, the vertical angles at H are equal. The angle at G is equal to the angle at B and the angle at E is equal to the angle at A. These triangles are similar and

$$\frac{HF}{AH} = \frac{GF}{AB} = \frac{1}{3};$$

$$HF = \frac{AH}{3};$$

$$4\,HF = AF;\ AF = \frac{HF}{4}.$$

The center of gravity of a triangular pyramid is on a line joining the vertex with the intersection of the medians of the base, at a distance from the base equal to one-fourth the length of this line. If h is the vertical distance of the vertex from the base, the center of gravity lines in a plane parallel to the base at a distance of $\frac{h}{4}$ therefrom.

A pyramid with quadrilateral base may be divided into two triangular pyramids. Since the center of gravity of each of these triangular pyramids is at a distance of one-fourth the vertical height from the base, the center of gravity of the entire pyramid lies in the same plane. This may be extended to pyramids of any form.

A cone may be regarded as made up of an infinite number of small pyramids of the same altitude. Its center of gravity is, therefore, at one-fourth the height from the base.

Problems

1. A homogeneous solid right cylinder, 5 inches in diameter and 6 inches long, stands with its axis vertical. On the top of the cylinder is placed a homogeneous solid right cone, which is 6 inches in diameter and 8 inches high. The axis of the cone is 1 inch from the axis of the cylinder. The density of cone and cylinder is the same. Find the center of gravity of the combination. *Ans.* $\bar{x} = 0.39$ in. from the axis of cylinder;
$\bar{y} = 4.95$ in. from the base of cylinder.

2. A homogeneous solid right cylinder is 10 inches in diameter and 8 inches high. A cone 6 inches in diameter and 6 inches deep is cut out of the top of the cylinder. The axis of the cone is 1 inch from the axis of the cylinder. Find the center of gravity of the remainder.

3. Solve Problem 1 if the density of the cylinder is twice the density of the cone.

4. A frustrum of a cone is 6 inches in diameter at the lower base, 4 inches in diameter at the upper base, and 6 inches high. Find its center of gravity.

100. Center of Gravity by Integration.—The calculation of the center of gravity is equivalent to the location of the line of application of the resultant of parallel forces. The moment of each mass is found with respect to some plane; then this moment is divided by the sum of the masses. If m_1 is the mass of one element of the body, and x_1 is its abscissa, its moment about any line in the YZ plane is m_1x_1. If m_1, m_2, m_3, etc. are masses, and x_1, x_2, x_3, etc., are the coördinates of their centers of gravity with respect to some plane of reference, the total moment with respect to that plane is

$$M = m_1x_1 + m_2x_2 + m_3x_3 + \text{etc.} = \Sigma mx. \quad (1)$$

$$\text{Total mass} = m_1 + m_2 + m_3 + \text{etc.} = \Sigma m. \quad (2)$$

$$\bar{x} = \frac{\Sigma mx}{\Sigma m}. \quad (3)$$

Similarly, the other coördinates of the center of gravity are

$$\bar{y} = \frac{\Sigma my}{\Sigma m}; \quad (4)$$

$$\bar{z} = \frac{\Sigma mz}{\Sigma m}. \quad (5)$$

In order to use these equations, the center of gravity of each of the parts which compose the system must be known. When the center of gravity of each part is not known, it is necessary to divide the body into infinitesimal elements, write the expression for the moment of each element, and find the sum of the moments by integration. If dm is the mass of an element and x' is the coördinate of the center of gravity, Equation (1) becomes

$$M = \int x'dm. \quad (6)$$

If ρ is the density and dV is the element of volume,

$$dm = \rho dV,$$

$$M = \int \rho x'dV. \quad (7)$$

If ρ is the mass of a plate per unit area, and dA is an element of area,

$$dm = \rho dA,$$

$$M = \int \rho x'dA.$$

If the density is uniform throughout all parts of the body, it may be omitted in finding the center of gravity.

With the integral expressions for the moment,

$$\bar{x} = \frac{\int x'dm}{\int dm} = \frac{\int \rho x'dV}{\int \rho dV}. \quad (8)$$

When the density is constant, Equation (8) becomes

$$\bar{x} = \frac{\int x'dV}{\int dV} = \frac{\int x'dV}{V}. \tag{9}$$

For a plate of uniform thickness and density,

$$\bar{x} = \frac{\int x'dA}{\int dA} = \frac{\int x'dA}{A}. \tag{10}$$

Similar expressions give $\bar{y}$ and $\bar{z}$.

It must be remembered that x' is the distance of the *center of gravity of the element* from the plane of reference.

Example I

Find the center of gravity of a triangular plate of uniform thickness and density, by moments about an axis through the vertex parallel to the base. (Fig. 144.)

The element of area is y high and dx wide. This form of element is chosen because the entire triangle may be built up of strips of this kind and because its moment arm is easy to find. Neglecting the small triangle at the top of the element, its area is ydx. The moment arm of the element about the line AB through the vertex of the triangle is

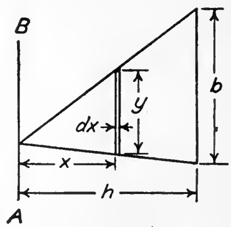


FIG. 144.

$$x' = x + \frac{dx}{2}.$$

When $\frac{dx}{2}$ is multiplied by the element of area, the product is a differential of the second order and may be neglected.

$$M = \int x'dA = \int xydx. \tag{11}$$

This expression for the moment contains two variables, x and y. One of these variables must be eliminated. From the geometry of the similar triangles of Fig. 144,

$$\frac{x}{y} = \frac{b}{h},$$

$$y = \frac{bx}{h}.$$

Substituting in Equation (11) and integrating,

$$M = \int \frac{bx^2}{h} dx = \frac{b}{h} \int x^2 dx = \frac{b}{h} \left[\frac{x^3}{3}\right]. \tag{12}$$

The limits for x are $x = 0$ and $x = h$,

$$M = \frac{b}{h} \left[\frac{x^3}{3}\right]_0^h = \frac{bh^2}{3}.$$

Dividing the moment by the area,

$$\bar{x} = \frac{bh^2}{3} \div \frac{bh}{2} = \frac{2h}{3}.$$

Problem

1. Instead of eliminating y from Equation (11) of Example I, eliminate x and dx and integrate between the proper limits for the moment. Also integrate ydx for the area of the triangle.

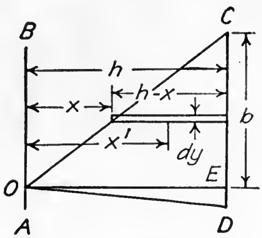


FIG. 145.

Example II

Find the center of gravity of a triangle by means of elements perpendicular to the base.

In Fig. 145, the vertical line CD is taken as the base of the triangle. With the origin at the vertex, x is the abscissa of a point on the line OC. The length of an element of area is $h - x$ and the width is dy

$$dA = (h - x)dy.$$

The center of gravity of the element is at the middle of its length, which is at a distance of $\frac{h - x}{2}$ from either end.

$$x' = x + \frac{h - x}{2} = \frac{h + x}{2}. \tag{13}$$

$$M = \int x'dA = \int \frac{(h + x)(h - x)dy}{2} = \int \frac{h^2 - x^2}{2} dy. \tag{14}$$

Since $y = \frac{bx}{h}$, $dy = \frac{b}{h} dx$. Substituting in Equation (14)

$$M = \frac{b}{2h} \int (h^2 - x^2)dx = \frac{b}{2h}\left[h^2x - \frac{x^3}{3}\right]_0^h = \frac{bh^2}{3}.$$

This result applies to the right triangle OCE. Dividing by the area, the center of gravity of this right triangle is found to be at two-thirds the altitude from the vertex. In a similar way the center of gravity of the other right triangle OED is found to be at the same distance from the vertex. Since the center of gravity of the entire triangle is on the line which joins these two centers, it is located at two-thirds the altitude from the vertex.

Problems

2. Substitute for x^2 in Equation (14) and integrate in terms of $\bar{x}$ as the independent variable.

3. Find the center of gravity of a quadrant of a uniform circular plate of radius a.

Using the vertical element of Fig. 146 to find the moment with respect to the plane of OA,

$$M = \int xydx = \int \left(a - x^2\right)^{\frac{1}{2}} xdx = \left[\frac{(a^2 - x^2)^{\frac{3}{2}}}{3}\right]_0^a = \frac{a^3}{3}.$$

$$\bar{x} = \frac{a^3}{3} \div \frac{\pi a^2}{4} = \frac{4a}{3\pi}.$$

Solve for $\bar{y}$ by means of the same element.

4. Find the center of gravity of a 60-degree sector of a circle of radius a by polar coördinates with double integration.

The element of area is $rd\theta dr$, and the moment arm for finding x is $x' = r \cos \theta$

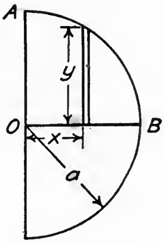


FIG. 146.

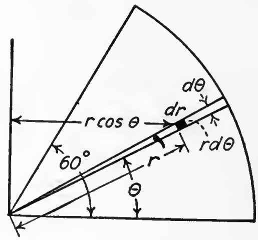


FIG. 147.

$$M = \int\int r^2 \cos \theta \, d\theta \, dr = \frac{a^3}{3} \int \cos \theta \, d\theta = \frac{a^3}{3}\Big[\sin \theta\Big]_0^{\frac{\pi}{3}}.$$

$$M = \frac{a^3\sqrt{3}}{6}.$$

$$\bar{x} = \frac{a^3\sqrt{3}}{6} \div \frac{\pi a^2}{6} = \frac{a\sqrt{3}}{\pi}.$$

The Y coördinate of the center of gravity may be found by a similar method, and may be checked by the condition that the center of gravity lies on the bisector of the angle.

5. Find the center of gravity of a 60-degree sector of a circle of radius a, using the bisector of the angle as the axis of X. Check trigonometrically with the results of Problem 4.

6. Find the center of gravity of a wire of uniform section which is bent into the form of a semicircle of radius a. Use polar coördinates with a single integraion.

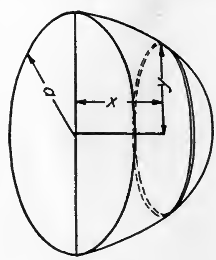


FIG 148.

$$Ans. \quad \bar{x} = \frac{2a}{\pi}; \ \bar{y} = 0.$$

7. Find the center of gravity of a homogeneous solid hemisphere. Build the hemisphere of circular disks as in Fig. 148.

$$Ans. \quad M = \pi \int (a^2 - x^2) x dx = \frac{\pi a^4}{4}$$

$$\bar{x} = \frac{\pi a^4}{4} \div \frac{2\pi a^3}{3} = \frac{3a}{8}.$$

8. A homogeneous solid cylinder, 8 inches in diameter and 6 inches long has a hemispherical depression, 6 inches in diameter, in one end. Find the center of gravity of the remainder.

9. Find the center of gravity of any cone of altitude h. Build up the cone of flat disks parallel to the base.

10. Find the center of gravity of a right cone by integration. Build up the cone of concentric hollow cylinders as in Fig. 149.

11. Find the center of gravity of a hemisphere by integration, using hollow cylinders as elements of volume.

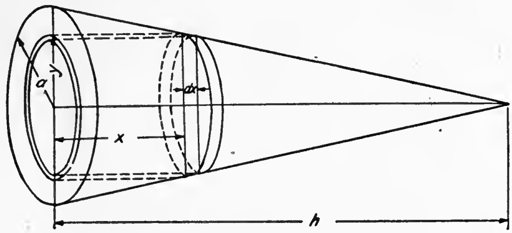


FIG. 149.

Example III

Find the center of gravity of a segment of a homogeneous sphere of radius a cut off by a plane at a distance of $\frac{a}{2}$ from the center.

In this case the volume of the body is not known and must be obtained by integration. Using an element in the form of a circular disk, as in Fig. 148, $dV = y^2dx$, and $x' = x$.

$$\bar{x} = \frac{\pi\int xy^2dx}{\pi\int y^2dx} = \frac{\pi\int(a^2 - x^2)xdx}{\pi\int(a^2 - x^2)dx} = \frac{\pi\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_{\frac{a}{2}}^{a}}{\pi\left[a^2x - \frac{x^3}{3}\right]_{\frac{a}{2}}^{a}};$$

$$\bar{x} = \frac{\pi a^4[\frac{1}{2} - \frac{1}{4} - (\frac{1}{8} - \frac{1}{64})]}{\pi a^3[1 - \frac{1}{3} - (\frac{1}{2} - \frac{1}{24})]} = \frac{\frac{9}{64}\pi a^4}{\frac{5}{24}\pi a^3} = \frac{27}{40}a.$$

The factor π in the numerator and denominator might have been canceled at the beginning. It is carried through, however, in order to have the true moment and volume at the end.

Do not make the mistake of canceling terms to the right of the integral signs. After integrating, common factors in the numerator and denominator may be canceled before the limits are put in, *provided one limit is zero.* It is safer, however, not to cancel any terms until after the limits are put in.

Problems

12. Make an approximate check of Example III, by considering a segment to be replaced by a cone.

13. Find the center of gravity of a uniform plate in the form of a segment of a circle of radius a which is cut off by a plane at a distance of $\frac{a}{3}$ from the center. Use a vertical element as in Fig. 150. When integrating for the area, substitute $x = a \cos \theta$, $y = a \sin \theta$.

Ans. $\bar{x} = 0.610a$; $\bar{y} = 0$.

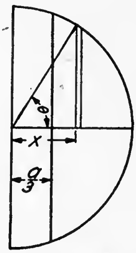


FIG. 150.

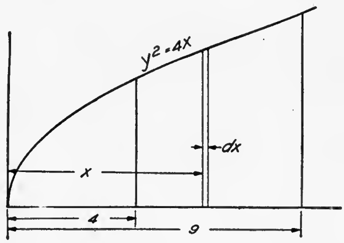


FIG. 151.

14. Find $\bar{y}$ for half the segment of Problem 13. Use the area already obtained for the entire segment and integrate for the moment only.

15. A uniform plate is bounded by the X axis, the curve $y^2 = 4x$ and the lines $x = 4$ and $x = 9$. Find its center of gravity, Fig. 151.

Ans. $\bar{x} = 6.66$; $\bar{y} = 2.57$.

16. Check Problem 15 approximately by replacing the area by the enclosed trapezoid.

17. Find $\bar{x}$ for the area bounded by the X axis, the line $x = a$, and the curve $y^2 = bx$. *Ans.* $\bar{x} = 0.6a$.

18. Find $\bar{x}$ for the area bounded by the X axis, the line $x = a$, and the line $y = bx$.

Ans. $\bar{x} = \frac{2a}{3}$.

19. Find $\bar{x}$ for the area bounded by the X axis, the line $x = a$, and the curve $y = bx^2$.

Ans. $\bar{x} = \frac{3a}{4}$.

20. Find $\bar{x}$ for the area bounded by the X axis, the line $x = a$, and the curve $y = bx^3$.

Ans. $\bar{x} = \frac{4a}{5}$.

21. A solid cylinder is cut by two planes which intersect on a diameter. One of these planes is perpendicular to the cylinder and the other makes an angle α with this plane, Fig. 152. Find the center of gravity of the part of the cylinder between the planes.

Ans. $\bar{x} = \frac{3\pi a}{16}$.

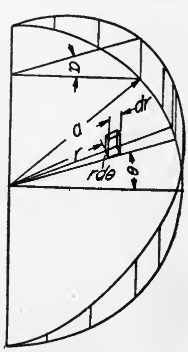


FIG. 152.

22. A solid cylinder of radius a is cut by two planes which intersect on a tangent and make an angle α with each other. One plane is normal to the axis of the cylinder. Find the center of gravity of the portion of the cylinder between the planes. Solve by polar coördinates.

101. Combination Methods of Calculation.—It frequently happens that part of an area or volume may be calculated by arithmetical methods, while the remainder requires the use of the calculus.

Example

Solve Problem 15 of Art. 100 with horizontal elements of area, Fig. 153.

There are two sets of elements. The elements of one set extend from $x = 4$ to $x = 9$; the elements of the other set, from the curve to $x = 9$. The elements of the first set form a rectangle so that it is not necessary to integrate for the area and moment. The area of this rectangle is 20

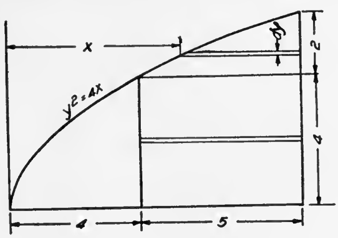


FIG. 153.

$$\bar{y} = \frac{20 \times 2 + \int \left(9 - \frac{y^2}{4}\right) y dy}{20 + \int \left(9 - \frac{y^2}{4}\right) dy};$$

$$\bar{y} = \frac{40 + \left[\frac{9y^2}{2} - \frac{y^4}{16}\right]_4^6}{20 + \left[9y - \frac{y^3}{3}\right]_4^6} = \frac{65}{\frac{76}{3}} = 2.57.$$

Problems

1. Find the center of gravity of the area bounded by the X and Y axes the curve $xy = 48$, the line $x = 12$, and the line $y = 8$.

Ans. $\bar{x} = 5.32$; $\bar{y} = ?$

2. Find the center of gravity of the area enclosed between the curve $xy = 48$ and a circle of radius 10 with its center at the origin.

3. Find the center of gravity of the area enclosed by the curve $y^2 = 4x$, the line $y = 2$, and the line $x = 9$.

102. Center of Gravity of Some Plane Areas.—The so-called center of gravity of a plane area is a factor of great importance in the study of Strength of Materials. A plane area is equivalent to a plate of uniform thickness and density.

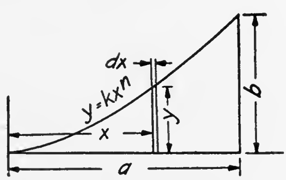


FIG. 154.

Figure 154 shows a plane area bounded by the X axis, the curve $y = kx^n$, and the ordinate $x = a$. The length of the extreme ordinate, where $x = a$, is $y = ka^n = b$.

To find the area with a vertical element,

$$dA = kx^n dx;$$

$$A = \left[\frac{kx^{n+1}}{n+1}\right]_0^a = \frac{ka^{n+1}}{n+1} = \frac{ab}{n+1}. \quad (1)$$

In a triangle, $n = 1$; $A = \frac{ab}{2}$.

In a parabola with the axis horizontal, $n = \frac{1}{2}$; $A = \frac{2ab}{3}$.

In a parabola with axis vertical, $n = 2$; $A = \frac{ab}{3}$.

To find the moment with respect to the YZ plane,

$$M = \int x'dA = k\int x^{n+1}dx = \frac{\left[kx^{n+2}\right]_0^a}{n+2}; \tag{2}$$

$$M = \frac{ka^{n+2}}{n+2} = \frac{a^2b}{n+2}. \tag{3}$$

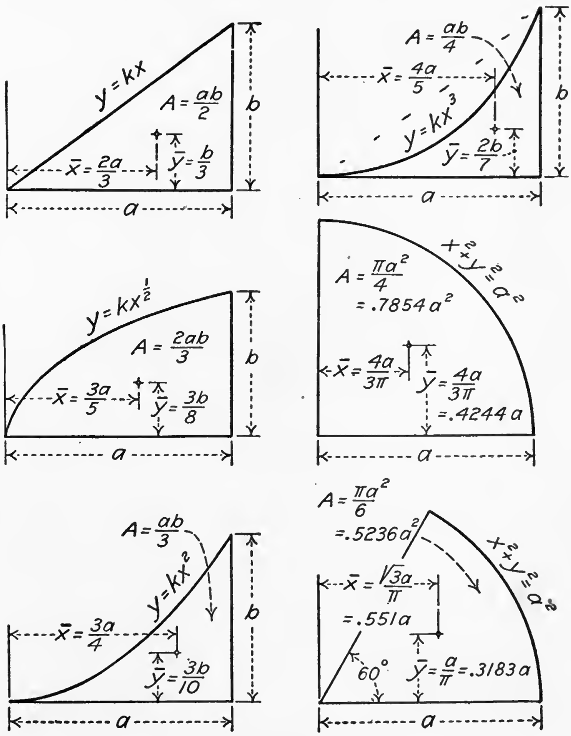


FIG. 155.

$$\bar{x} = \frac{M}{A} = \frac{a^2b}{n+2} \div \frac{ab}{n+1} = \frac{(n+1)a}{n+2}. \tag{4}$$

In a triangle, where $n = 1$, $\bar{x} = \frac{2a}{3}$.

In a parabola with axis horizontal, where $n = \frac{1}{2}$, $\bar{x} = \frac{3a}{5}$.

In a parabola with axis vertical, where $n = 2$, $\bar{x} = \frac{3a}{4}$.

To find the moment with respect to the XZ plane,

$$M = \int \frac{y}{2}\,dA = \frac{k^2}{2}\int x^{2n}dx = \frac{k^2}{2}\frac{[x^{2n+1}]_0^a}{2n+1}; \tag{5}$$

$$M = \frac{k^2a^{2n+1}}{2(2n+1)} = \frac{ab^2}{2(2n+1)}. \tag{6}$$

$$\bar{y} = \frac{M}{A} = \frac{ab^2}{2(2n+1)} \div \frac{ab}{n+1} = \frac{(n+1)b}{2(2n+1)}. \tag{7}$$

In a triangle, where $n = 1$, $\bar{y} = \frac{b}{3}$.

In a parabola with axis horizontal, where $n = \frac{1}{2}$, $\bar{y} = \frac{3b}{8}$.

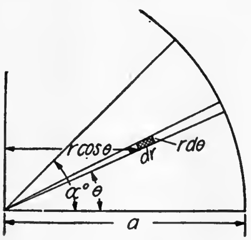


FIG. 156.

Figure 155 shows four of the most important cases of areas bounded by the X axis, the ordinate $x = a$, and the curve $y = kx^n$. It shows also the center of gravity of a 90-degree sector and a 60-degree sector.

Figure 156 shows any sector of a circle of radius a bounded on one side by the X axis, and on the other by a radius at an angle with the X axis. The area is πa^2 multiplied by the ratio of the arc to the circumference of the circle. If $\alpha°$ be the angle in degrees, $A = \frac{\pi a^2 \alpha°}{360}$.

To find the moment with respect to the YZ plane with polar coördinates,

$$M = \int r^2 \cos\theta\, d\theta\, dr = \frac{a^3}{3}\int \cos\theta\, d\theta = \frac{a^3}{3}[\sin\theta]_0^{\alpha°}. \tag{8}$$

$$\bar{x} = \frac{M}{A} = \frac{a^3}{3}\sin\alpha° \div \frac{\pi a^2\alpha°}{360} = \frac{120\, a \sin\alpha°}{\pi°}. \tag{9}$$

For a 60-degree sector, $\sin\alpha = \frac{\sqrt{3}}{2}$, and

$$\bar{x} = \frac{120a\frac{\sqrt{3}}{2}}{60\pi} = \frac{\sqrt{3}a}{\pi}.$$

Since these sectors are symmetrical with respect to the radius at angle $\frac{\alpha}{2}$ with the X axis, $\bar{y}$ may be calculated by multiplying $\bar{x}$ by the tangent of the half-angle.

Problems

1. By means of Equation (9), find the center of gravity of a 45-degree sector.

2. By means of the answer of Problem 1 find the center of gravity of a sector of which the angle is 22° 30′.

3. Find the center of gravity of the area bounded by the X axis, the curve $y = x^{\frac{1}{3}}$, and the line $x = 8$ by means of the equations of this article.

4. Find the center of gravity of a segment cut off by a line at a distance from the center equal to one-half the radius. Calculate the area and moment of the half sector and subtract the moment and area of the triangle.

5. Find the center of gravity of the area bounded by the X axis, the curve $y^2 = 4x$, and the lines $x = 4$ and $x = 9$. Find the moment and area of the entire area to the line $x = 9$ and then subtract for the area to the left of $x = 4$.

103. Liquid Pressure.—The pressure of a liquid is the same in all directions, and is proportional to the depth below the surface. If w is the weight of unit volume of the liquid, and y the depth of the surface below the surface of the liquid, the pressure per unit area is wy. The density of water is nearly 62.5 pounds per cubic foot; the constant w for water is 62.5 when foot and pound units are used. With gram and centimeter units, $w = 1$.

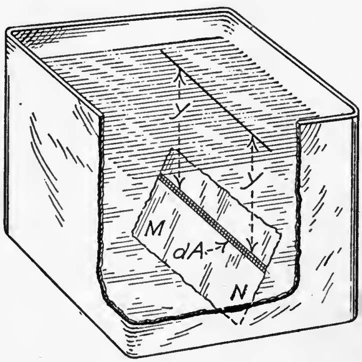


FIG. 157.

Figure 157 represents a tank of liquid in which there is a submerged surface MN. An element of the surface, dA, is located at a vertical distance y below the surface of the liquid. The total pressure on one side of the surface on the element dA is $wydA$. The total pressure on the entire surface is

$$\text{total pressure} = \int wy\, dA = w\int ydA. \tag{1}$$

Since $\int ydA$ is the moment of the area dA with respect to the surface of the liquid

$$\int ydA = \bar{y}A, \tag{2}$$

$$\text{total pressure} = w\bar{y}A. \tag{3}$$

The total pressure of a liquid on a surface is the weight of a column of liquid whose area is the area of the surface and whose height is the depth of the center of gravity of the surface below the surface of the liquid.

Problems

1. One end of the tank of Fig. 157 is 6 feet wide and 8 feet high to the surface of the water. Find the total horizontal pressure on this vertical surface. *Ans.* 12,000 lb.

2. A vertical gate, 5 feet wide and 4 feet high, is subjected to pressure of water which rises 6 feet above the top of the gate. Find the total horizontal pressure on the gate. *Ans.* 10,000 lb.

3. A cylindrical tank with axis horizontal is 6 feet in diameter. It is filled with water and connected to a 4-inch pipe in which water stands 8 feet above the axis of the cylinder. Find the total pressure on the end. *Ans.* 14,137 lb.

4. Solve Problem 3 if the cylinder is half full of water. *Ans.* 1125 lb.

5. A tank with triangular ends is 6 feet wide at the top and 4 feet high. Find the pressure at one end when the tank is filled with water. If the tank is 6 feet long, find the pressure on one side.

104. Center of Pressure.—In the case of a vertical or inclined surface, the pressure varies with the depth. The *average* pressure

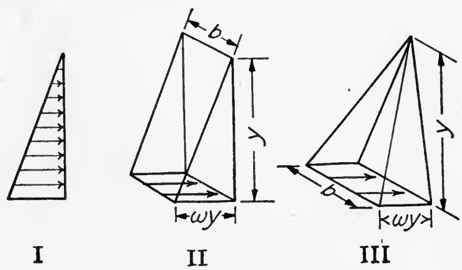


Fig. 158.

is that at the center of gravity of the area. The *resultant* pressure is below the center of gravity. Fig. 158 shows the variation of pressure against a vertical surface. The same relation holds for an inclined surface. Since the pressure varies as the depth below the surface, it may be represented by the weight of a volume of liquid between two planes which intersect at the surface of the liquid. In the case of a rectangular plane surface, Fig. 158, II, the pressure is equivalent to the weight of a wedge of constant width equal to the width of the surface. The pressure at the bottom is wy. The average pressure is $\frac{wy}{2}$, which is the pressure at the center of gravity of the area. The resultant pressure is located at the center of gravity of the wedge. The center of gravity of the wedge is the same as that of a triangular plate, which is two-thirds the height from the vertex. The position of the resultant pressure is called the *center of pressure.* The center of pressure on a rectangular surface which reaches the top of the liquid is two-thirds the height of the surface below the surface of the liquid. This distance is measured parallel to the rectangular surface.

Problems

1. A vertical rectangular gate is 5 feet wide and 6 feet high. It is subjected to pressure of water which just reaches the top. Find the total pressure and the center of pressure. If the gate is held by bolts at the top and bottom, find the tension in each.

Ans. Tension in top bolts = 1875 lb.; tension in bottom bolts = 3750 lb.

2. A vertical rectangular gate is 5 feet wide and 12 feet high. It is subjected to the pressure of 12 feet of water on one side and the pressure of 6 feet of water on the other. Find the force at the top and bottom required to hold the gate against this pressure. *Ans.* 6562 lb. at the top.

3. A vertical triangular gate is 8 feet wide at the bottom and 12 feet high. It is subjected to water pressure on one side. Find the center of pressure, Fig. 158, III.

4. A vertical rectangular gate, 8 feet wide and 12 feet high, is subjected to the pressure of water which rises 4 feet above the top. Find the horizontal force at the top required to hold the gate. Where is the center of pressure? Fig. 159.

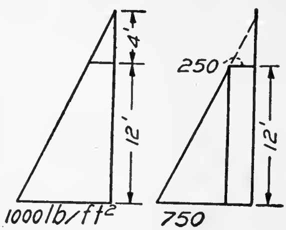


FIG. 159.

5. Find the center of pressure of the semicircular area of Problem 4 of the preceding article.

Generally, the location of the center of gravity of the wedge-shaped solid which represents the pressure is not known. In Art. 153, a general method of solution is given, which involves the radius of gyration of the area of the surface. The determination of the center of pressure by means of the wedge of pressure is convenient for rectangular surfaces.

105. Summary.—The resultant of parallel forces in space is equal to the algebraic sum of the forces. The moment of the resultant about any axis is equal to the sum of the moments of the separate forces about that axis.

For a problem of equilibrium of parallel forces in space there are three unknowns and three independent equations. These equations may be two moments and one resolution or three moments.

The center of mass is the point of application of the resultant of parallel forces which are proportional to the masses of the particles upon which they act. Center of mass is practically the same as center of gravity.

Center of gravity is calculated by dividing the sum of the moments by the sum of the masses. This calculation locates one plane through the center of gravity. Two other planes are found in the same way. The center of gravity lies at the intersection of the three planes.

$$\bar{x} = \frac{\int x'dm}{\int dm},$$

$$\bar{y} = \frac{\int y'dm}{\int dm},$$

$$\bar{z} = \frac{\int z'dm}{\int dm},$$

in which $\bar{x}$, $\bar{y}$ and $\bar{z}$ are coördinates of the center of gravity of the body and x', y', and z' are the coördinates of the elements or particles which compose the body.

When the density is constant, the elements of volume may replace the elements of mass. When the body is a plate of uniform thickness and density, the elements of area may be used in place of elements of mass.

If a body of uniform density has a plane of symmetry, the center of gravity lies in that plane.

The center of gravity of a triangular plate of uniform density and thickness is located at the intersection of the medians, which is at a distance from the base equal to one-third of the altitude of the triangle.

The center of a pyramid or cone of uniform density is located at a distance from the base equal to one-fourth the altitude.

The pressure of a liquid on a plane surface consists of parallel forces perpendicular to the surface. This pressure is directly proportional to the vertical depth below the surface of the liquid. The average pressure on any surface is the pressure at the center of gravity of the surface. The point of application of the resultant pressure is called the center of pressure. The center of pressure may be found by means of the center of gravity of a wedge-shaped solid. The center of pressure on a rectangular plane surface which extends to the surface of the liquid is at two-thirds the altitude of the surface below the surface of the liquid.

CHAPTER XI

FORCES IN ANY POSITION AND DIRECTION

106. Couples in Parallel Planes.—In Art. 70, it was shown that a couple may be balanced by another couple of equal magnitude and opposite sign acting in the same plane, and that the forces of this second couple might have any magnitude, direction, and position in the plane. It will now be shown that it is not necessary for the couples to be in the same plane. A couple may be balanced by a couple in any parallel plane, provided the moments are equal and opposite. Figure 160 shows a wheel to which two ropes are attached. The ropes run horizontally in opposite directions. If a force P is applied to the lower rope, and if the upper rope is attached to a fixed point, the tensions in the two ropes will be equal and opposite, and form a counter-clockwise couple. The center of gravity of the system is supposed to be located at **W.** The plane of the weight and of the upward reaction R is parallel to the plane of the wheel. It is possible to make the force **P** of such magnitude that the system is in equilibrium. The weight **W** and the reaction **R** form a clockwise couple, which is equal and opposite to that of the forces in the ropes.

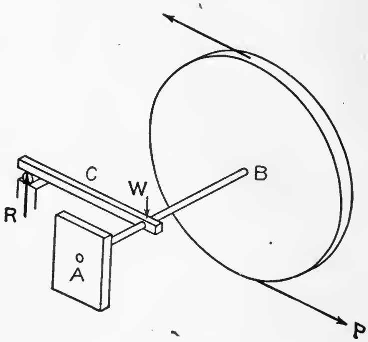


Fig. 160.

Figure 161 shows two forces, $\mathbf{P}_1$ and $\mathbf{P}_2$, each of magnitude **P.** These forces are at a distance a apart in the same vertical plane. Suppose that an opposite couple made up of two forces $\mathbf{Q}_1$ and $\mathbf{Q}_2$ at a distance b apart acts in a parallel plane. If the moments are equal so that

$$Pa = Qb$$

it may be shown that these couples balance and produce equilibrium.

According to Art. 70, the couple Qb may be replaced by an equal couple Pa in its plane. In Fig. 161, these forces are $\mathbf{P}_3$ and $\mathbf{P}_4$ and are parallel to $\mathbf{P}_1$ and $\mathbf{P}_2$. In Fig. 161, $ABCD$ is a quadrilateral perpendicular to the forces. Since $AB = a = CD$, and since AB is parallel to CD, the quadrilateral is a parallelogram. The upward force $\mathbf{P}_1$ of the first couple and the upward force $\mathbf{P}_3$ of the second couple may be replaced by a force of magnitude **2P** at a point midway between them. The downward force $\mathbf{P}_2$ and the downward force $\mathbf{P}_4$ may be replaced by their resultant of magnitude **2P** at a point midway between C and B. The line of the upward resultant passes through the middle of one diagonal of the parallelogram, and the line of the downward force passes through the middle of the other diagonal. These two forces must lie in the same line. They are equal and opposite; consequently, they balance and produce equilibrium.

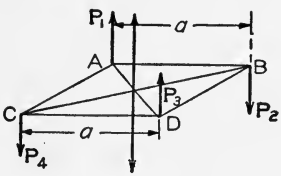


Fig. 161.

Since two couples which balance the same couple are equivalent, two couples in parallel planes which are numerically equal and have the same sign are equivalent. The combined moment of two couples in the same plane is the algebraic sum of the moments of the separate couples. The combined moment of couples in parallel planes is the algebraic sum of the moments of the several couples.

The addition of couples in parallel planes may be proved in a slightly different way. In Fig. 161, suppose that both couples are clockwise so that the force $\mathbf{P}_4$ is up and the force $\mathbf{P}_3$ is down. The resultant of $\mathbf{P}_1$ and $\mathbf{P}_4$ is a force **2P** midway between them. The resultant of $\mathbf{P}_2$ and $\mathbf{P}_3$ is an equal downward force. These two resultants are at a distance a apart, and form a couple of magnitude $2Pa$. This couple is in a plane parallel to the planes of the original couples and midway between them.

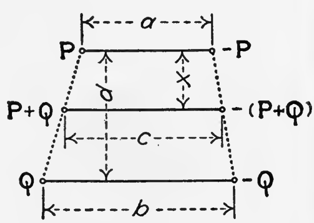


Fig. 162.

Figure 162 applies to unequal couples. It will be shown that the resultant couple may be in any plane parallel to the plane of the couples to be added. This figure represents a plane perpendicular to the planes of the two couples. One of these couples is made up of a force **P** upward perpendicular to the plane of the paper and an equal force downward at a distance a from this upward force. In the second couple the forces are **Q** downward at the right and an equal force upward at a distance b. The distance between the planes of the two couples is d. The resultant of the upward parallel forces **P** and **Q** is a force **P + Q** at a distance x from the plane of the first couple. By moments

$$x = \frac{Qd}{P + Q}. \qquad (1)$$

The resultant of the two downward forces is a downward force **P + Q** at the same distance from the plane of the first couple. The moment arm of the resultant couple is c.

$$c = a + \frac{x}{d}(b - a) = a + \frac{(b-a)\,Q}{P + Q}; \qquad (2)$$

$$c = \frac{Pa + Qb}{P + Q}. \qquad (3)$$

$$\text{Resultant moment} = (P + Q)c = Pa + Qb. \qquad (4)$$

The couple Qb may be replaced by any other couple in its plane, provided the moment is the same in magnitude and sign. If **Q** were made larger and b smaller, the plane of the resultant couple would be closer to the couple Qb. By changing the relative values of **Q** and b, it may be shown that the resultant couple may come in any plane parallel to the planes of the original couples.

An ordinary windlass affords an example of the equilibrium of parallel couples. The force on the crank and part of the reactions of the bearings form one couple. The load on the rope and the remainder of the reactions of the bearings form the other couple. This arrangement, and the similar one of pulleys on shafting are complicated by the fact that the reactions at the bearings are not usually in the same plane as the turning forces, and must be resolved into components to get a pair of couples.

107. A Couple as a Vector.—It has been shown that a couple may be replaced by another couple of the same magnitude and sign. The forces of the couple may have any magnitude and position, and may be located anywhere in the plane of the original couple or in any plane parallel to the plane of the original couple.

The easiest way to express the direction of a plane is by means of the direction of a line normal to it. All parallel planes have the same normal. A couple may be represented by a line normal to its plane. The length of the line gives the magnitude of the couple. The direction of the line bears the same relation to the direction of rotation of the couple as the direction of motion of a right-handed screw bears to its direction of rotation. A clockwise couple is represented by a vector away from the observer, a counter-clockwise couple by a vector toward the observer.

A vector *partly* represents a force, since it gives its direction and magnitude, but does not give its location. A vector *completely* represents a couple, since the forces of the couple may be anywhere in a given plane or in parallel planes.

That a force may be represented by a vector, and that the resultant of two forces is given by their vector sum has been accepted as an axiom which has been amply verified by experiment. If two couples which are not in parallel planes be represented by vectors, and if it may be shown that the resultant of these two couples is a third couple which is represented fully by the vector sum of these vectors, then the assumption that a couple is a vector may be regarded as valid. *This* proof will be given in Art. 108.

108. Resultant Couple.—Figure 163, I, represents two planes which intersect on the line AB. A counter-clockwise couple of moment Pa is supposed to be acting in the plane CAB and a second counter-clockwise couple Qb in the plane DAB. The couple Pa may be anywhere in its plane. One force $\mathbf{P}_2$ may be regarded as acting along the line AB and the second force $\mathbf{P}_1$ at a distance a from that line. The couple Qb may be replaced by a couple Pc in its plane, provided $Pc = Qb$. One force of this new couple may be regarded as acting in the line AB in a direction opposite to the force $\mathbf{P}_2$. The remaining force will then be in the plane DAB at a distance c from the line AB. The forces $\mathbf{P}_2$ and $\mathbf{P}_4$ acting along the line AB in opposite directions balance each other. The remaining forces $\mathbf{P}_1$ and $\mathbf{P}_4$ form a new counter-clockwise couple. If d is the distance from $\mathbf{P}_1$ to $\mathbf{P}_4$, the moment of this couple is Pd. Figure 163, II, is a space triangle in a plane perpendicular to the planes CAB and DAB. The line HL of length d in this triangle is the distance between the forces of the resultant couple. Figure 163, III, is a vector diagram. The line EF of length Pa represents the original couple in the plane CAB.

The line EF is normal to this plane, and therefore, lies in the plane of the triangle HLK, and is perpendicular to the line HK. The line FG is normal to the plane DAB and is normal to the line KL of the space triangle. Since

$$\frac{Pa}{Pc} = \frac{a}{c},$$

and since the angle at F is equal to the angle at K, the space diagram and the vector diagram are similar triangles. As the

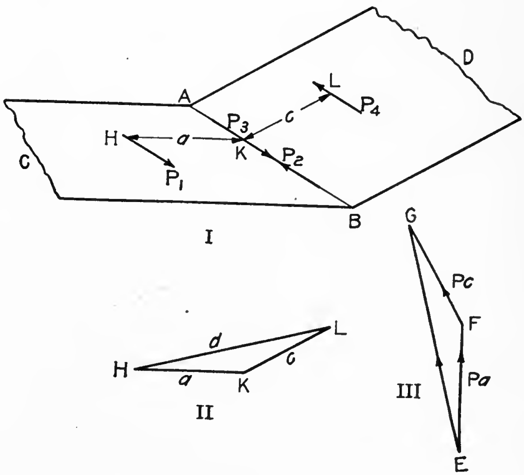


FIG. 163.

angle at E is equal to the angle at H, EG is normal to HL. Since the sides are proportional

$$\frac{EG}{Pa} = \frac{d}{a},$$
$$EG = Pd$$

If two couples are represented by vectors normal to their respective planes, the resultant of these couples is represented by the vector sum of the two vectors.

Example

A force of 12 pounds is vertically downward. Another force of 12 pounds vertically upward is located 2 feet east of this first force. A second couple is made up of a horizontal force of 8 pounds directed north and an equal force directed south at a distance of 4 feet above this force. Find the resultant couple.

The first force has a moment of 24 foot-pounds. It is represented by a horizontal vector 24 units in length directed south. The second couple is represented by a horizontal vector 32 units in length directed east. The resultant couple is represented by their vector sum. This is a horizontal vector 40 units in length directed south 53° 08′ east. The resultant couple is in a vertical plane normal to this resultant vector.

Problem

1. A box is 6 feet long from east to west, 4 feet wide, and 3 feet high. It weighs 160 pounds and its center of mass is at its center. At the northwest corner at the bottom, the box rests upon a support which permits it to turn but not to move laterally. A horizontal force of 60 pounds directed south is applied at the south east corner at the bottom. A horizontal force of 80 pounds directed east is applied at the northwest corner at the top. In what plane will these three forces and the reactions at the support turn the box, and what is the resultant moment?

When the planes of the couples which are to be combined are not at right angles to each other, it is advisable to resolve each vector along three coördinate axes; then combine the components in exactly the same way as was used in the calculation of the resultant of non-coplanar forces.

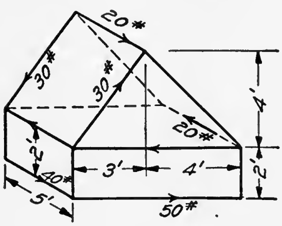


FIG. 164.

Problem

2. In Fig. 164, find the sum of the components of the moment vectors along each of the three axes.

Ans. 170 ft.-lb. about a vertical axis; 120 ft.-lb. about a horizontal axis toward the left; 150 ft.-lb. about a horizontal axis toward the front.

109. Forces Reduced to a Force and a Couple.—In Chapter VI it was shown that a set of non-concurrent, coplanar forces may be reduced to a single force and a single couple in the same plane. It will now be shown that a set of non-concurrent forces, which are not coplanar, may likewise be reduced to a single force and a single couple. The force and the couple will not, in general, be in the same plane.

In Fig. 165, the line AB represents the line of a single force of magnitude **P**. This force may be replaced by an equivalent force **P** at some point O and a couple in the plane AOB. The moment of this couple is the product of the force **P** multiplied by the distance from O to its line of action. A second force **Q** (not shown in the drawing may likewise be replaced by a force **Q** at O and a couple in the plane through O and its line of action. The resolution of any number of forces in this way, gives a set of concurrent forces at O and a set of couples. The concurrent, non-coplanar forces at O may be combined into a single resultant

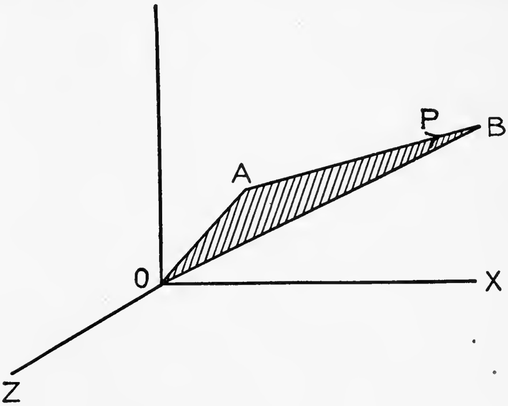


FIG. 165.

by the methods of Art. 89. The couples may be combined into a single couple by a similar addition of vectors.

In the case of *non-concurrent, coplanar* forces, the single force and the single couple may be further reduced to a single force, except when the resultant force is zero. When the resultant force is zero, the final result is a couple. In the case of *non-concurrent, non-coplanar* forces, the force and the couple can not, in general, be combined into a single force. In Fig. 166, **P** represents a force through the point O, and Qb represents a couple in a different plane. One of the forces of the couple may be made to intersect the force **P** at O. By changing the value of b, the force $\mathbf{Q}_1$ may be given any desired magnitude. If, however, the

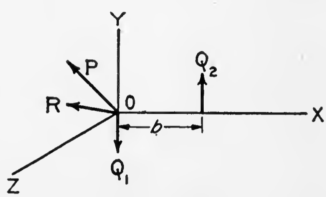


FIG. 166.

couple does not lie in a plane parallel to the direction of the force **P**, the two intersecting forces can not be made to balance. In general, the resultant of the forces **P** and $\mathbf{Q}_1$ will be a force **R**. This force **R** is not in the same plane with the force $\mathbf{Q}_2$. The set of forces reduces to a force and a couple, and the force and the couple reduce to two forces, which are not in the same plane and do not intersect.

By changing the distance b, the value of **Q** is changed, hence the *magnitude* and *position* of the force $\mathbf{Q}_2$ and the *magnitude* and *direction* of the force **R** may be changed.

110. Equilibrium of Non-concurrent, Non-coplanar Forces.— It has been shown that a set of non-concurrent, non-coplanar forces may be reduced to a single force through some arbitrarily chosen point and a single couple. The magnitude and direction of the force are determined by the conditions of the problem. The magnitude and direction of the couple depend upon the conditions of the problem and the location of the arbitrarily chosen point through which the single force acts. To specify fully the single force through a known point, *three* factors are required. These may be the magnitude of the force and two angles, or the components along each of three axes. To specify fully the couple as a vector, *three* factors are required. These may be one magnitude and two directions, or three components. A problem of non-concurrent, non-coplanar forces in equilibrium may, therefore, involve *six* unknowns and require six equations for its solution.

For equilibrium, the resultant force must be zero. The force diagram for all the forces at the arbitrarily chosen point forms a closed polygon in space. Each force at this arbitrary point is identical in magnitude and direction with the force which it replaces. If all the non-concurrent forces of the problem are regarded as concurrent and combined into a single force polygon, this polygon is closed. The sum of the components along each of three axes is zero.

The resultant couple must also be equal to zero. The forces and reactions on the rigid body combine to make up this resultant couple. In order that the couple may be zero, it is necessary that the sum of the moments about any axis be equal to zero.

To solve a problem of equilibrium in which there are six unknowns, it is necessary to write six independent equations. Three of these must be moment equations. The remaining three may be either resolution equations or moment equations. It is advisable to begin with a moment equation about some axis which will eliminate most of the unknowns.

In the special instance in which all the forces are parallel, the number of unknowns is reduced to three. Since the forces are all in the same direction, only one resolution equation may be written. The other two equations must be moment equations.

If all the forces are in parallel planes, the resolution perpendicular to these planes will give zero for each force. In this case there may be only two resolutions. The total number of unknowns is reduced to five.

Example

A horizontal trap door, Fig. 167, is 8 feet long and 6 feet wide. It is supported by two hinges, each 1 foot from a corner, on an 8-foot edge. It is lifted by a rope attached to the other 8-foot edge at a distance of 2 feet

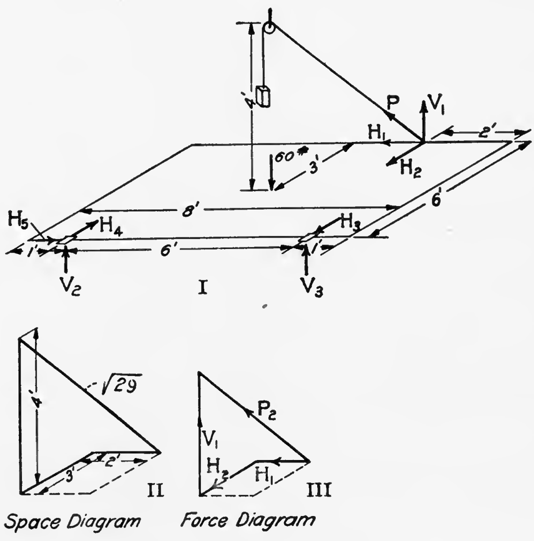


FIG. 167.

from one corner. This rope passes over a smooth pulley which is 4 feet above the center of the door. The door weighs 60 pounds and its center of mass is at the center. Find the tension in the rope and the horizontal and vertical components of the hinge reaction, assuming that the hinges are so constructed that the horizontal force parallel to their line is all taken by the left hinge.

The unknown quantities are the tension in the rope, the horizontal and vertical components of the reaction at the right hinge, two horizontal components and one vertical component at the left hinge. The tension of the rope is made up of three components. Since the direction of the rope is known, the problem is fully solved when one of these components is determined.

A moment equation about the line of the hinges eliminates five of the unknowns, together with the horizontal components of the tension of the rope. The forces which have moment about this line are the weight of the door and the vertical component of the tension in the rope.

$$60 \times 3 = V_1 \times 6,$$
$$V_1 = 30 \text{ lb.}$$

The tension in the rope and its horizontal components may now be computed by resolutions. (These resolutions form no part of the six equations, since the door is not under consideration as the free body.) The space diagram, Fig. 167, II, is similar to the force diagram, Fig. 167, III. All the dimensions of the space diagram are known. The vertical force $\mathbf{V}_1$ of the force diagram is also known. The other forces may be found from the geometry of the similar solids. If **P** is the tension in the rope,

$$\frac{P}{30} = \frac{\sqrt{29}}{4};$$
$$P = 40.39 \text{ lb.}$$

Similarly, $H_1 = 15$ lb., $H_2 = 22.5$ lb. These components are convenient in finding the components at the hinges.

Taking moments about an axis which passes through the left hinge and is parallel to the 6-foot edges,

$$V_3 \times 6 = 60 \times 3 - 30 \times 5;$$
$$V_3 = 5 \text{ lb.}$$

Taking moments about an axis through the right hinge parallel to the 6-foot edges,

$$V_2 \times 6 = 60 \times 3 - 30 \times 1;$$
$$V_2 = 25 \text{ lb.}$$

Check by a vertical resolution,

$$V_1 + V_2 + V_3 = 30 + 25 + 5 = 60.$$

To find $\mathbf{H}_4$, take moments about a vertical axis through the right hinge. This eliminates $\mathbf{H}_3$ and $\mathbf{H}_5$.

$$H_4 \times 6 = H_1 \times 6 + H_2 \times 1$$
$$15 \times 6 = 90$$
$$22.5 \times 1 = 22.5$$
$$6H_4 = 112.5,$$
$$H_4 = 18.75 \text{ lb.}$$

To find $\mathbf{H}_3$, take moments about a vertical axis through the left hinge,

$$15 \times 6 = 90 \text{ counter-clockwise,}$$
$$22.5 \times 5 = 112.5 \text{ clockwise,}$$
$$6H_3 = 22.5 \text{ clockwise,}$$
$$H_3 = 3.75 \text{ lb.}$$

The moment of $\mathbf{H}_3$ about the vertical axis through the left hinge must balance a clockwise moment. The direction of $\mathbf{H}_3$ must be opposite the direction of the arrow in Fig. 167. $\mathbf{H}_3$ might have been computed by a horizontal resolution. Resolving as a check,

$$3.75 + 18.75 = 22.5.$$

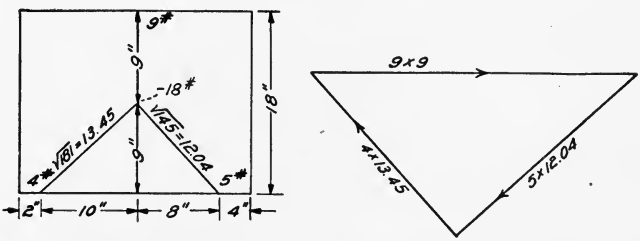
By resolutions parallel to the line of the hinges,

$$H_5 = H_1 = 15 \text{ lb.}$$

The problem has been solved by five moment equations and one resolution equation, and partly checked by two resolution equations. Three of the moments were taken about horizontal axes and two were taken about vertical axes.

Problems

1. A horizontal trap door, similar to Fig. 167, is 10 feet long, 7 feet wide, weighs 80 pounds, and has its center of mass at the center. It is hinged 1 foot from the corners on one long side and lifted by a rope at one corner of the opposite side. The rope passes over a smooth pulley which is 8 feet above the center of the door. Find all the reactions.



Space Diagram *Vector Diagram of Couples*

FIG. 168.

2. A horizontal plank, 24 inches long, 18 inches wide, weighs 18 pounds and has its center of gravity at the center. It is supported at the middle of one 24-inch side, and 2 inches from one corner and 4 inches from the other corner of the other 24-inch side. Find the reactions of these supports.

Ans. 9 lb., 4 lb., 5 lb.

3. In problem 2 consider the downward force of 18 pounds as made up of 9 pounds, 4 pounds, and 5 pounds, and consider that these forces form couples with the three reactions. The vectors which represent these couples lie in the plane of the board. Draw the vector diagram for the couples, Fig. 168.

4. In Fig. 168, the reaction of 9 pounds is moved 4 inches to the right of the middle. Find the other reactions and draw the vector diagram for the three couples in equilibrium.

5. The mast of a stiff-leg derrick, Fig. 169, is 30 feet high. The stiff legs are in vertical planes at right angles to each other. One of these planes extends south and the other extends west of the mast. The stiff legs make angles of 45 degrees with the vertical. The boom is 40 feet long and carries a load of 2,000 pounds on the end. Find the tension in the stiff legs and the reactions at the bottom of the mast resulting from this load when the boom is elevated 20 degrees above the horizontal in a vertical plane which is north 50 degrees east.

Regard the boom, mast, and the cables which connect them as a free body in equilibrium. Begin by taking moments about a horizontal north and south line through the bottom of the mast. This eliminates the reactions at the bottom of the mast and the tension in the stiff leg in the north and south vertical plane. Let $\mathbf{V}_1$ be the vertical component of the tension in the west stiff leg. Consider this component applied at the base of the leg, where the moment arm of the horizontal component is zero.

$$V_1 \times 30 = 2000 \times 40 \cos 20° \cos 40° = 57{,}585 \text{ ft.-lb.}$$
$$V_1 = 1919 \text{ lb.}$$

The tension in the west stiff leg $= V_1 \sec 45° = 2714$ lb.

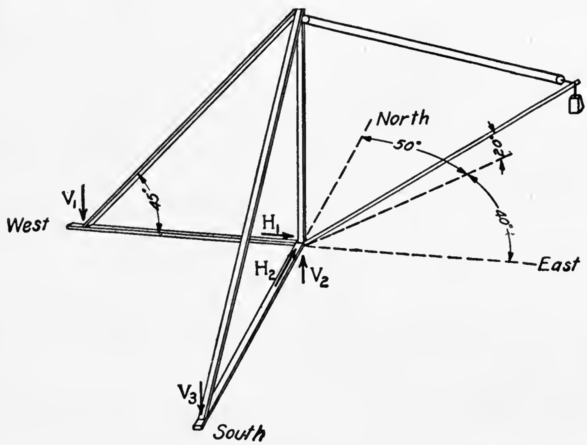


FIG. 169.

Find the tension in the other stiff leg by moments in a similar manner and find the three components of the reaction at the base by three resolutions. Check by moments.

There are only five unknowns. The only forces which are not in the plane of the mast and boom are the forces which act on the mast. If there were transverse forces acting on the boom, there could be another reaction with a horizontal component. This would be the sixth unknown.

6. Solve Problem 5 when the boom is south 10 degrees east. What is the direction of the vertical reaction at the bottom of the mast?

The student should look up a stiff-leg derrick. What provision is made for the reversed reaction at the bottom of the mast when the boom is longer than the horizontal projection of the legs? What provision is made for resisting the vertical component of the tension in the legs?

7. A derrick mast is 40 feet long. The boom is 50 feet long, weighs 1000 pounds, and has its center of mass at the center. One guy rope is west and another is north 10 degrees east. These ropes make angles of 30 degrees with the horizontal. The boom is horizontal in a position south 60 degrees east and carries a load of 2000 pounds. Find the tension in the west guy rope by moments about a horizontal line through the bottom of the mast and the bottom of the other guy rope. *Ans.* 3443 lb.

8. At what position of the boom in Problem 7 will the tension in the west guy rope be the greatest, and what will be its value?

Ans. 3664 lb. when the boom is south 80 degrees east.

9. Find the four remaining unknowns in Problem 7.

10. A windlass, Fig. 170, is 1 foot in diameter and 4 feet long between bearings. The crank is 2 feet long from the center of the crank pin to the axis of the axle. The force is applied to the crank pin at a distance of 1 foot from the plane through the right bearing perpendicular to the axis. A rope wound round the windlass 20 inches from the left bearing carries a load of 480 pounds. The crank makes an angle of 30 degrees to the right of the

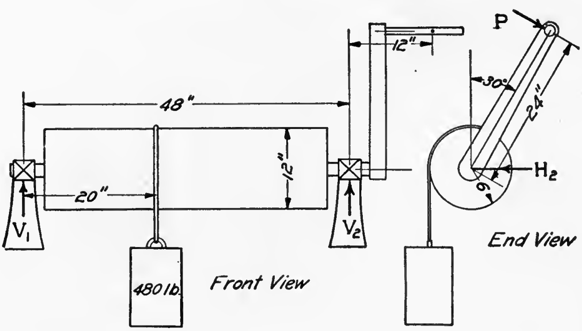


FIG. 170.

vertical upward viewed from the right. The force on the crank is normal to the plane through the axis of the windlass and the crank pin. Find this force and the horizontal and vertical reactions at the bearings. The forces are all in planes perpendicular to the axis so that there are only five unknowns.

Ans. P = 120 lb.; V_1 = 265 lb.; V_2 = 275 lb.; H_1 = 25.98 lb.; H_2 = 129.90 lb.

11. Solve Problem 10 when the crank is 45 degrees to the right of the vertical.

12. A shaft is 6 feet long from center to center of bearings and weighs 200 pounds. At one foot from the left bearing it carries a pulley 4 feet in diameter which weighs 240 pounds. At 2 feet from the right bearing it carries a pulley 3 feet in diameter which weighs 180 pounds. The belt from the left pulley runs forward horizontally and exerts a pull of 600 pounds at the top and 150 pounds at the bottom. The belt from the right pulley passes around a pulley 6 feet in diameter on a second shaft. The axis of this second shaft is 10 feet below the axis of the first shaft. The tension in the front of this belt is 200 pounds. Find the tension in the other part of the belt and the components of the reactions of the bearings.

13. A door is 10 feet high, 12 feet wide, weighs 180 pounds, and has its center of gravity at the center. It is hinged 1 foot from the top and 1 foot from the bottom. The top hinge is 6 inches north of a vertical line through the bottom hinge. The hinges are so placed that the lower hinge takes all the force parallel to the edge of the door. Find the force on each hinge when the door is directly south of the hinges. The door is turned until the top and bottom are east and west and is held by a horizontal force applied to the edge 4 feet from the bottom. Find this force and the components of the forces of the hinges.

14. A rod 12 feet long, weighing 270 pounds, with its center of mass at the center, is hung horizontally north and south and supported by three cords. One cord is attached 2 feet from the north end. The others are attached 2 feet from the south end. One of the latter makes an angle of 34 degrees within the vertical in a vertical plane which is north 75 degrees east. The other makes an angle of 45 degrees with the vertical in a vertical plane which is south 60 degrees west. Find the direction of the cord near the north end and the tension in each of the three.

Ans. 6° 34′ north of the vertical, tension 136.9 lb.; 92.9 lb.; 81.9 lb.

15. A bar, weighing 114 pounds, with its center of mass at the middle, is supported in a north and south horizontal position by means of three ropes. One rope is attached 5 feet north of the middle. The second rope is attached 2 feet south of the middle. The upper end of this rope is fastened to a point which is 5 feet above, 2 feet south and 3 feet east of the point of attachment to the bar. The third rope is attached 7 feet south of the middle of the bar. The upper end of this rope is fastened to a point which is 6 feet above, 4 feet west and 1 foot north of the point of attachment to the bar. Find the tension in each rope and the direction of the first one.

Ans. First rope is 19° 02′ with the vertical in a vertical plane which is north 38° 40′ west. The tensions are 51.44 lb., 52.84 lb., and 27.30 lb.

16. A wheelbarrow is 5 feet long from axle to handles, and 2 feet wide between handles. A load of 180 pounds is placed 3 inches to the right of the center and 12 inches from the axis of the axle. The wheelbarrow is held horizontal and pushed up an inclined plane which makes an angle of 5 degrees with the horizontal. Find the reaction at the wheel and the direction and magnitude of the force at each handle.

17. A cylinder 6 inches in diameter and weighing 60 pounds is suspended in a horizontal position by means of two ropes. One rope supports the cylinder 2 feet from the middle. This rope passes under the cylinder, and both ends leave the cylinder in the same vertical plane perpendicular to its axis. One end makes an angle of 45 degrees with the vertical and the other makes an angle of 30 degrees with the vertical. The friction is sufficient to prevent the cylinder from slipping. The second rope is wound round the cylinder and fastened to it. How far must this second rope be placed from the middle of the cylinder? Solve graphically, neglecting the diameter of the ropes.

18. A right cone, 2 feet in diameter at the base and 8 feet high, is supported with its axis horizontal by means of a vertical rope which is wound round it several times but is not fastened to it. The friction is sufficient to prevent slipping. One end of the rope leaves the cone 4 inches from the base. Where must the other end leave? Solve graphically.

111. Summary.—Couples in parallel planes which have the same magnitude and sign are equivalent. If their signs are opposite, they balance each other. Couples in parallel planes are added algebraically.

A couple may be represented by a vector perpendicular to its plane. The magnitude of the vector is proportional to the moment of the couple. The direction of the vector bears the same relation to the direction of rotation of the couple as the direction of motion of a right-handed screw bears to its direction of rotation. (This is merely a convention. The opposite direction might have been chosen just as well.)

A set of non-concurrent, non-coplanar forces acting on a rigid body may be reduced to a single force at any arbitrarily chosen point and a single couple. This single force is equivalent to the resultant of all the forces which act on the body. The force and the couple may be reduced to two forces. These two forces do not, in general, intersect.

Any rigid body may be held in equilibrium by a single force which is equal and opposite to the resultant of the applied forces and a couple. Any rigid body may be held in equilibrium by two forces of proper magnitude, direction, and location.

The conditions of equilibrium are:

(1) The force polygon must close.

(2) The sum of the moments about any axis must be zero.

The condition that the force polygon must close is expressed algebraically by the statement that the sum of the components along any axis is equal to zero. By taking resolutions along three axes, not more than two of which are in the same plane, three independent equations are obtained. By taking moments about three axes, three more independent equations are obtained.

A problem of the equilibrium of non-concurrent, non-coplanar forces may involve six unknowns and six independent equations of mechanics. There may be other equations depending upon the geometric, algebraic, or other conditions of the problem. These six equations may be three resolutions and three moments. Any or all of the resolutions may be replaced by moment equations.

Many problems involve fewer than six unknowns. In Problem 2 of Art. 110, all the forces are parallel. One resolution completely specifies these forces, hence two resolution equations must be omitted. The forces exert no moment about a vertical axis.

This eliminates one possible moment equation. When non-coplanar forces are parallel, the number of unknowns is reduced to three. In Problem 5 of Art. 110, there is no moment about a vertical axis. This eliminates one couple and reduces the possible unknowns to five. In problem 10 of Art. 110, the forces have no components parallel to the axis of the cylinder. This eliminates a resolution equation and reduces the unknowns to five. In Problem 13, the forces are coplanar in the first position and there are three unknowns. In the second position, the problem involves six unknowns In Problem 14, one couple is left out and the unknowns are reduced to five.

CHAPTER XII

FRICTION

112. Coefficient of Friction.—Figure 171 represents a body of W pounds mass on a horizontal plane. A rope runs horizontally from the body, passes over a smooth pulley, and supports a mass of P pounds. When the load P is relatively small, the system remains stationary. Regarding the mass W as the free body and resolving horizontally, the tension $\mathbf{P}'$ in the horizontal cord (which is nearly equal to the load P) is balanced by the horizontal force from the supporting plane at the surface of contact. This horizontal force is the friction between the two bodies at their surface of contact. *Friction is a force exerted between two bodies at their surface of contact.* The force of friction resists motion of one body parallel to the surface of the other. In Fig. 171, the tension $\mathbf{P}'$ tends to pull the mass W toward the right. The friction from the plane to the body acts toward the left as is indicated by the arrow.

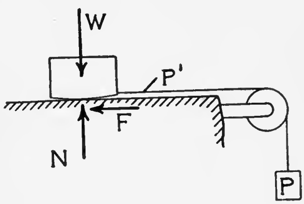


FIG. 171.

If the load P is gradually increased, the tension $\mathbf{P}'$ will finally reach a value greater than the friction. The mass W will begin to move toward the right and continue to move with increasing speed. If an additional load be applied to the mass W, the force required to start it in motion will also be increased. The body W may be a wooden block weighing 10 pounds and the force required to start it may be 4 pounds. If an additional 10-pound weight be put on the block, the force required to start it will be about 8 pounds.

The force which must be overcome in order to start a body in motion is called the *static* or *starting friction.*

If a force is sufficient to start a body in motion, that force will cause the body to move with increasing speed after it has been started. A considerably smaller force will keep it in motion with uniform speed, after it has been started. If the force $\mathbf{P}'$ is gradually increased, a value will be found at which the body will continue to move with constant speed after it has been set in motion by an additional force. This force which will keep the body in motion with constant speed after it has been started is somewhat less than the force required to start it. The resistance when the body is moving at constant speed is called the *moving friction*, the *sliding friction*, or the *kinetic friction*.

The ratio of the friction to the pressure normal to the surface is called the coefficient of friction.

$$f = \frac{F}{N}, \quad F = fN, \qquad \text{Formula X}$$

in which f is the coefficient of friction, F is frictional resistance, and N is the normal pressure.

Example

In Fig. 171, the mass W weighs 12 pounds and the load P required to start it in motion is 5.4 pounds. The 'oad P required to keep the body in motion after it has been started is 4.2 pounds. Find the coefficient of moving friction and the coefficient of starting friction.

If the friction of the pulley is negligible, it may be assumed that the horizontal tension in the cord is equal to the weight P. For the coefficient of starting friction,

$$f_0 = \frac{5.4}{12} = 0.45.$$

For the coefficient of moving friction,

$$f = \frac{4.2}{12} = 0.35$$

The coefficient of starting friction is very irregular. For moderate loads, the coefficient of moving friction between surfaces which are not lubricated is fairly constant. For very low pressures, and for very high pressures which exceed the elastic limit of the material, the coefficient of moving friction is greater than it is for moderate pressures. Friction depends upon the material and upon the polish of the surfaces of contact.

Friction is greatly reduced by lubrication with suitable oils. If the speed is sufficient to keep a good oil film between the surfaces, the friction is that of solids on viscous liquids, and the coefficient of friction decreases with increase of pressure.

In the problems below it will be assumed that the coefficient of friction is constant.

Problems

1. A 60-pound mass is pulled along a horizontal plane by a horizontal force of 21 pounds. Find the coefficient of friction. *Ans.* $f = 0.35$.

2. A 60-pound mass is pulled along a horizontal plane by a force of 20 pounds at an angle of 20 degrees above the horizontal. Find the coefficient of friction.

Ans. $f = \frac{18.79}{53.16} = 0.353.$

3. The coefficient of friction between a wooden block weighing 40 pounds and a horizontal wooden floor is 0.32. What horizontal force will keep the block moving with constant speed?

4. Solve Problem 3 if the force makes an angle of 20 degrees above the horizontal.

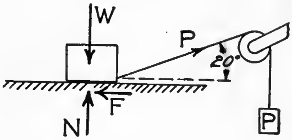


FIG. 172.

Resolving horizontally, Fig. 172,

$$P \cos 20° = F = 0.32\, N.$$

Resolving vertically,

$$N = 40 - P \sin 20°.$$

Ans. $P = 12.20$ lb.

5. A 60-pound mass on a plane inclined 25 degrees to the horizontal is just pulled up the plane by a force of 40 pounds acting upward parallel to the plane. Find the coefficient of friction. *Ans.* $f = \frac{14.64}{54.38} = 0.27.$

6. In Problem 5, what force upward, parallel to the plane, will allow the 60-pound mass to slide down the plane with uniform speed after it has once started? *Ans.* 10.72 lb.

7. In Problem 5, what horizontal force is required to keep the 60-pound mass moving up the plane with uniform speed? *Ans.* 50.5 lb.

8. A ladder 24 feet long rests on a horizontal floor and leans against a smooth vertical wall. The ladder makes an angle of 20 degrees with the vertical. The center of gravity of the ladder and its load is 16 feet from the bottom. What must be the minimum value of the coefficient of friction in order that the ladder may not slide down after it has been started by a slight jar?

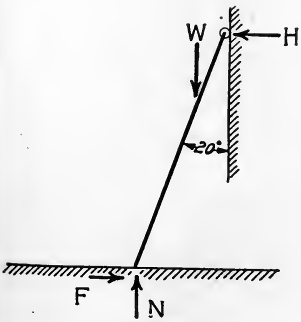


FIG. 173.

Solve by one moment and two resolution equations, together with Formula X. (Fig. 173.) *Ans.* $f = 0.24.$

9. Solve Problem 8 if the wall is not smooth but has a coefficient of friction equal to that of the horizontal floor.

10. A ladder weighing 40 pounds, with its center of gravity 12 feet from the lower end, rests on a horizontal floor and leans over the upper edge of a smooth vertical wall at a distance of 20 feet from the floor. The ladder makes an angle of 25 degrees with the vertical. The coefficient of friction at the floor is 0.35. How far up the ladder may a man weighing 160 pounds climb? *Ans.* 18.7 ft.

11. A 10-pound mass is placed on a 30-degree inclined plane. If the coefficient of moving friction is 0.1, what force parallel to the plane will just pull the body up the plane? What force will permit the body to slide down with uniform speed? *Ans.* 5.866 lb.; 4.13 lb.

12. Solve Problem 11 if the forces are horizontal. *Ans.* 7.19 lb.; 4.52 lb.

113. Angle of Friction.—In the space diagram of Fig. 174, a body of W pounds mass is placed on an inclined plane which makes an angle ϕ with the horizontal. The component of the weight down the plane is $W \sin \phi$. The component normal to the plane is $W \cos \phi$. If this body slides down the plane with uniform speed after it has once been started, the component of its weight parallel to the plane is equal to the moving friction.

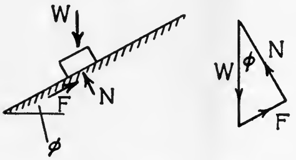


FIG. 174.

$$F = W \sin \phi. \tag{1}$$

$$f = \frac{W \sin \phi}{W \cos \phi} = \tan \phi. \tag{2}$$

The angle at which a plane must be inclined to the horizontal in order that a body on it may just slide down with uniform speed is called the *angle of friction.* The angle at which a plane must be inclined in order that a body may *start* down the plane is called the angle of *starting* friction. If the inclination of a plane slightly exceeds the angle of sliding friction, a slight vibration will set the body in motion and it will continue to slide with increasing velocity.

Problems

1. In Problem 1 of Art. 112, what is the angle of friction?
Ans. $\phi = 19° 17'$.

2. A 40-pound mass is placed on a plane which makes an angle of 40 degrees with the horizontal. It is found that a pull of 8 pounds up the plane is required in order that the body may slide down the plane with uniform speed. Find the coefficient of friction and the angle of friction.

The force diagram of Fig. 174 shows that the resultant of the force of friction and the normal force is a vertical force which is equal and opposite to the weight of the body. If the inclination of the plane were less than the angle of friction, the force triangle would be similar to this figure. The resultant of friction and the normal reaction would be vertical and equal to the weight. The friction, however, would be less than its limiting value and the body would remain stationary.

In Fig. 175, four forces are in equilibrium. The resultant of **F** and **N** makes an angle ϕ with the normal to the surface. In all cases where a body is sliding with uniform speed there is a force from the surface of contact to the moving body, which is the resultant of the normal reaction and the moving friction.

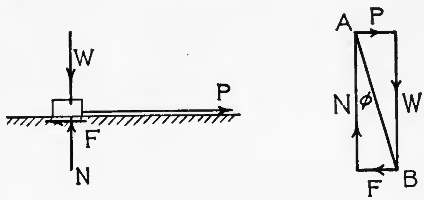


FIG. 175.

The angle which this resultant force makes with the direction of the normal to the surface is the angle of friction. If the body is at rest at the condition of incipient sliding, the tangent of the angle which the resultant of the friction and the normal forces makes with the normal is the coefficient of starting friction.

Example

A body is moved along a horizontal plane by a force which makes an angle θ with the horizontal. What is the value of θ in order that the force may be a minimum?

In Fig. 176, the weight W is laid off as a vertical line of known length. The resultant of the normal and the friction makes an angle ϕ with the vertical. The tangent of ϕ is the coefficient of friction. Through one end of the vector **W** of the force diagram, a line of indefinite length is drawn at an angle ϕ with the vertical. This line gives the direction of the resultant of the friction and the normal. A line from the other end of **W** to the line of this resultant represents the force **P**. The length of this line is the least when it is perpendicular to the line of the resultant. Since the resultant makes an angle ϕ with the vertical, the line **P** perpendicular to the resultant makes the same angle with the horizontal. The pull is a minimum when its angle with the horizontal plane is the angle of friction.

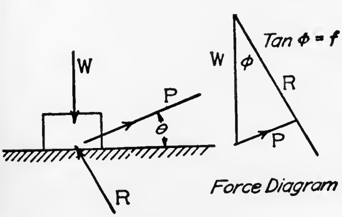


FIG. 176.

Problems

3. A body is pulled up an inclined plane by a force which makes an angle θ with the plane. Show graphically that the pull is the least when its angle with the plane is equal to the angle of friction.

4. Solve Problem 8 of the preceding article by means of the direction condition of equilibrium.

5. A body is drawn along a horizontal plane by a force which makes an angle with the plane equal to the angle of friction. Show that this force is the product of the weight of the body multiplied by the sine of the angle of friction.

When a body is stationary or moving with uniform speed along a surface, the resultant of the friction and the normal reaction of the surface is equal and opposite to the resultant of all the other forces which act on the body. If the body is moving with uniform speed, the resultant of the normal and the friction makes an angle with the normal whose tangent is the coefficient of

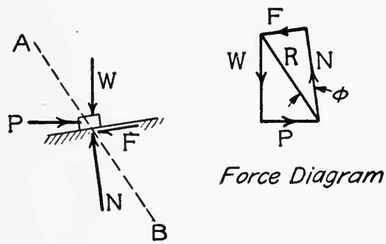


FIG. 177.

sliding friction. The resultant of all the other forces makes the same angle with the normal. In Fig. 177, the external forces are **W** and **P**. Their resultant is **R**. This resultant balances the resultant of the friction and the normal.

If ϕ is the angle of friction,

$$\tan \phi = \frac{F}{N}; \tag{3}$$

$$\sin \phi = \frac{F}{R} \tag{4}$$

In the case of lubricated surfaces, where the coefficient of friction is small, $\tan \phi$ is practically equal to $\sin \phi$, and the resultant **R** differs little from the normal **N**.

114. Cone of Friction.—Figure 178, I, shows a body pushed along a plane by a force which has a large component normal to the plane. The component against the plane greatly increases the normal pressure and, as a result, greatly increases the friction. Figure 178, II, is the force diagram. If the angle between the force **P** and the plane is further increased, the vectors **P** and **R** of Fig. 178, II, will become more nearly parallel, will intersect at greater distances, and will represent correspondingly greater forces.

In Fig. 178, III, the force **P** makes an angle with the vertical which is equal to the angle of friction. The force polygon will not close. It is impossible to move the body by a force in this direction. If the force makes a still smaller angle with the

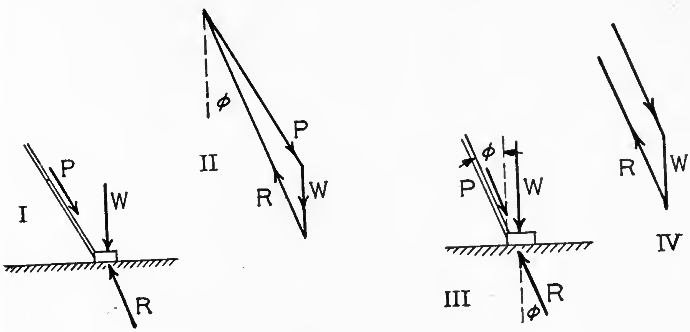


FIG. 178.

vertical, as shown in Fig. 178, IV, the force polygon will not close when the force is downward toward the surface. When a force which pushes a body against a surface makes an angle with the normal which is less than the angle of friction, this force will not move the body along the surface, no matter how great it may be. The lines which make the same angle with the normal to a surface form a cone, as shown in Fig. 179. When this angle is the angle of friction, the cone is called the cone of friction. A force which is applied along a direction inside the cone of friction for a body and surface has no tendency to move the body along the surface. The component of such a force normal to the surface increases the friction more than the tangential component increases the component parallel to the surface.

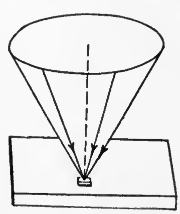


FIG. 179.

When a body is moved along a surface, the applied force must be outside the cone of friction for the body and the surface. A force, applied inside the direction of the cone of friction, tends to hold the body stationary.

115. Bearing Friction.—Figure 180 shows a shaft of radius *a* turning in a very loose bearing in a clockwise direction. No such bearing would be used in practice. It is drawn in this way in Fig. 180 in order to show details more clearly. When the shaft is turned, it climbs up on the bearing until it reaches a point at which it slides down as fast as it rolls up. At this point, the resultant pressure makes an angle with the normal which is equal to the angle of friction for the surfaces of the shaft and the bearing.

If there is a wheel rigidly attached to the shaft, and if forces **P** and **Q** are applied to this wheel by means of a belt, the resultant moment of these forces about the axis of the shaft must be equal to the moment of the friction about that axis. In Fig. 180, the

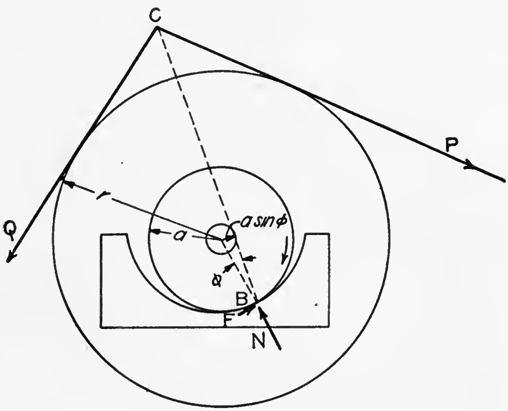


Fig. 180.

moment of the friction is counter-clockwise. The moment of the force **P** must then be greater than the moment of the force **Q**. The line of the normal at the point of contact B passes through the axis of the shaft. The resultant of the friction and the normal reaction of the bearing makes an angle ϕ with the direction of this normal. The resultant of the external forces **P** and **Q** passes through their intersection at C. If the weight of the wheel and of the shaft are neglected, the resultant of the forces **P** and **Q** will pass through B. The resultant of **P** and **Q** will be equal and opposite to the resultant of the normal reaction and the friction at B, and will lie along the same line.

If a circle of radius $a \sin \phi$ is drawn concentric with the axis of the shaft, the line of the resultant reaction at B will be tangent to this circle. This circle, called the *friction circle,* is convenient for some graphical solutions.

Example

In Fig. 181 the shaft is 6 inches in diameter. The wheel is 2 feet in diameter and weighs 200 pounds. The force **P** is 240 pounds. The force **Q** is 200 pounds. Both forces are vertically downward. Find the coefficient of friction.

Taking moments about the axis of the shaft,

$$3F = 240 \times 12 - 200 \times 12 = (240 - 200) \times 12 = 480 \text{ ft.-lb.}$$
$$F = 160 \text{ lb.}$$

Since all the applied forces are parallel, their resultant is equal to their sum. The resultant is 640 pounds vertically downward.

$$\text{Sin } \phi = \frac{F}{R} = \frac{160}{640} = 0.25;$$
$$\phi = 14° \ 29'.$$
$$f = \tan \phi = 0.258.$$

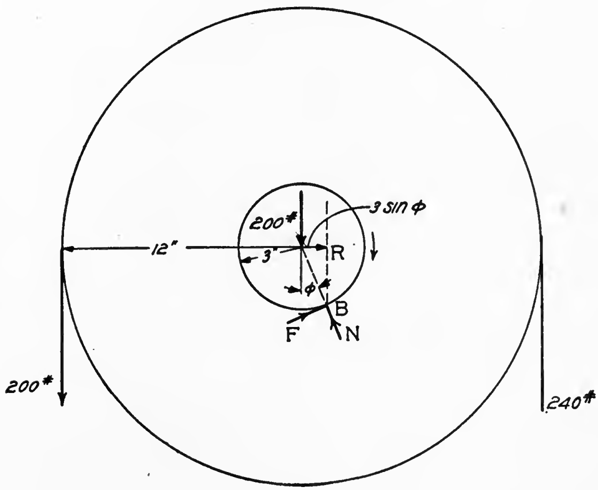


FIG. 181.

The student will understand that this bearing is not lubricated. The coefficient of friction for a well lubricated bearing should be many times smaller than 0.258.

Since all the applied forces are vertical, this problem may be conveniently solved by moments about the point of contact of the shaft and bearing. The equation of moments about this point B of Fig. 181 is

$$200(12 + 3 \sin \phi) + 200 \times 3 \sin \phi = 240(12 - 3 \sin \phi)$$

Problems

1. In the Example above, the force **Q** is vertical and the force **P** is horizontal toward the right. Find the coefficient of friction, and the position of the point of contact of the shaft and bearing.

Ans. $R = 467$ lb.; $f = 0.365$. The normal at the point of contact makes an angle of $20° \; 04' + 30° \; 58' = 51° \; 02'$ with the vertical.

2. An axle 4 inches in diameter is connected to a wheel 2 feet in diameter. The wheel and axle together weigh 160 pounds. A rope passes over the wheel and hangs vertically downward on each side. A load of 240 pounds is placed on one end of the rope. The coefficient of friction is 0.16. What is the load on the other end of the rope which will lift the 240-pound load?

Ans. 257.3 lb.

3. Solve Problem 2 if the axle is only 2 inches in diameter at the bearings.

4. Solve Problem 3 if the bearings are so well lubricated that the coefficient of friction is 0.03.

Approximate solutions of problems of bearing friction are often sufficiently accurate. One approximation is the assumption that the resultant is equal to the normal pressure. This is equivalent to $R = N$, and $\sin \phi = \tan \phi$. When the coefficient of friction is small this method involves little relative error. Another approximation is the calculation of the resultant (but not the moment) on the assumption that $P = Q$. In Problem 2, if it is assumed that $P = Q$ in the calculation of the resultant,

$$R = 240 + 240 + 160 = 640 \text{ lb.}$$

If it is assumed that $R = N$,

$$F = 0.16 \times 640 = 102.40 \text{ lb.}$$

Taking moments about the axis of the shaft,

$$(Q - 240) \times 12 = 102.40 \times 2;$$
$$(Q - 240) = 17.07 \text{ lb};$$
$$Q = 257.07 \text{ lb.}$$

It is not probable that the coefficient of friction is known with anything like the relative accuracy of this result.

Problems

5. Solve Problem 3 by the approximate methods.

6. An axle 4 inches in diameter is attached to a wheel 4 feet in diameter. The wheel and axle together weigh 200 pounds. A rope hangs vertically downward on one side of the wheel and supports a load of 300 pounds. On the other side of the wheel the rope makes an angle of 30 degrees below the horizontal. What is the tension in this part of the rope required to lift the load of 300 pounds if the coefficient of friction is 0.12? Solve by the approximate method.

116. Rolling Friction.—When a cylinder rolls on a surface, the surface is depressed somewhat by the pressure which the cylinder exerts on it. This is shown on an exaggerated scale in Fig. 182.

If the material of the surface is resilient, it will spring back so as to push against the cylinder on both sides of the lowest point. If its resilience is small, it will come back slowly so that practically all the pressure will be in front of the lowest point of the cylinder. In any case, the reaction of that part of the surface in front of the cylinder will be greater than the reaction on the other side, and the resultant of all the reactions will be somewhat as shown in the figures. The cylinder may be regarded as continuously climbing a small hill. Experiments have shown that the distance of the resultant reaction from the line of the load **W** is practically constant for a given material and is independent of the diameter of the cylinder. This distance, f_r of Fig. 182, is the coefficient of rolling friction. For steel on steel the coefficient of rolling friction is from 0.02 inch to 0.03 inch.

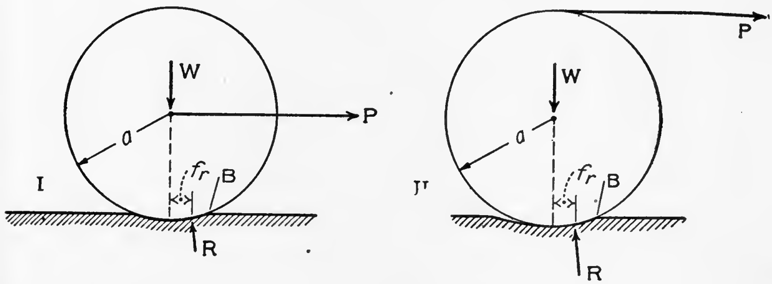


FIG. 182.

In Fig. 182, I, the cylinder is rolled along a horizontal surface by a horizontal force applied at the center. If moments are taken about the point of contact B, the moment arm of the force **P** is practically equal to the radius of the cylinder. The moment arm of the weight **W** is the coefficient of rolling friction f_r. The moment equation is then,

$$Wf_r = Pa. \tag{1}$$

In Fig. 182, II, the cylinder is rolled along by means of a force parallel to the surface applied at the top. The moment arm of this force is the diameter of the cylinder.

$$Wf_r = 2Pa. \tag{2}$$

Problems

1. If the coefficient of rolling friction of iron on iron is 0.03 inch, what horizontal force applied at the center of an iron wheel, 18 inches in diameter, weighing 200 pounds, will just roll it along a smooth, horizontal iron surface.

Ans. 0.67 lb.

2. An iron wheel, 30 inches in diameter, weighing 160 pounds, rolls on a steel rail. The coefficient of rolling friction is 0.03 inch. The wheel is attached to an axle 4 inches in diameter. A load of 800 pounds is applied to the axle by means of a bearing. The coefficient of sliding friction between the axle and the bearing is 0.04. What horizontal pull is required to move the load? *Ans.* 6.32 lb.

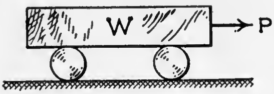


Fig. 183.

3. Figure 183 represents a mass of **W** pounds on cylindrical rollers. The coefficient of rolling friction is 0.04 inch at the bottom and 0.05 inch at the top. The cylinders are 10 inches in diameter and weigh 100 pounds each. The load **W** is 2000 pounds. If the track is horizontal, what is the force required to keep the load moving? *Ans.* 18.8 lb.

4. In Problem 3, what is the pull required to keep the load moving up a 1-degree inclined plane?

117. Roller Bearings.—Since the coefficient of rolling friction of hard steel on hard steel is very small, it is desirable to design bearings in which the friction is rolling rather than sliding. This is accomplished by means of *roller bearings*, or *ball bearings*. Fig. 184 shows a roller bearing. The axle rotates on hardened steel rollers which in turn roll on the inside surface of a hollow steel cylinder. In order to keep the rollers properly spaced so that they will not come together and introduce sliding friction, each roller is connected at the ends to a light metal frame called the cage.

For relatively light loads, ball bearings are employed instead of roller bearings. Spherical balls roll in grooves or races, which may be *V* shaped or circular.

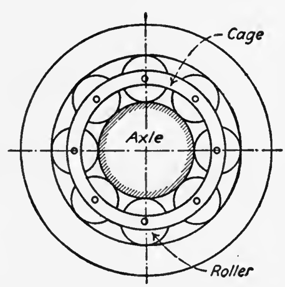


Fig. 184.

For rolling, the coefficient of starting friction is nearly the same as the coefficient of moving friction. This property is one of the elements of the superiority of roller or ball bearings over the ordinary bearings.

118. Belt Friction.—Figure 185, I, represents a pulley and a belt. The belt is in contact with the pulley through an angle of α radians. The tension of the belt at the left point of tangency with the pulley is $\mathbf{P}_1$. The tension at the right point of tangency is $\mathbf{P}_2$. The tension $\mathbf{P}_2$ is greater than $\mathbf{P}_1$ so that the belt tends to turn the pulley in a clockwise direction.

$$P_2 - P_1 = \textit{friction between belt and pulley.}$$

Figure 185, II, shows a small element of the length of the belt. This element is enclosed between radii which make an angle $d\theta$ with each other. The tension on the left end of the element is P; on the right end is $P + dP$. The difference between these two tensions is dP. This difference is equal to the friction between the belt and the pulley. Since the two tensions, P and $P + dP$,

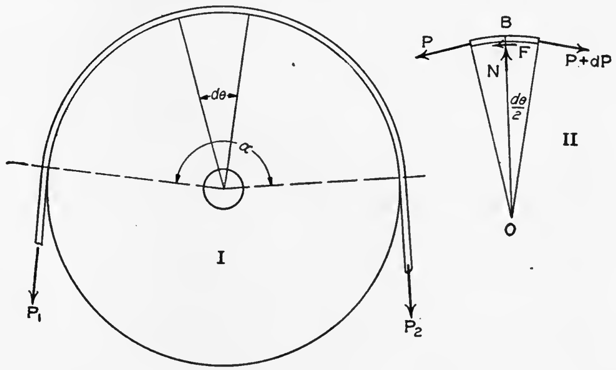


FIG. 185.

are each tangent to the pulley, they make an angle $d\theta$ with each other, and each tension makes an angle $\frac{d\theta}{2}$ with the tangent at the middle of the element. The normal reaction of the pulley on the element of the belt is obtained by a resolution along the radius BO. The component of P is $P \sin \frac{d\theta}{2}$, which, for a small angle, is equal to $P\frac{d\theta}{2}$. When the differentials of the second order are omitted, the component of $P + dP$ is also $P\frac{d\theta}{2}$. The total normal force on the element is

$$N = Pd\theta. \tag{1}$$

If the coefficient of friction between the belt and the pulley is f, then

$$F = fN = fPd\theta = dP. \tag{2}$$

Separating the variables in this last equation,

$$\frac{dP}{P} = f\,d\theta; \tag{3}$$

$$\log_e P = f\theta + C. \tag{4}$$

When $\theta = 0,\ P = P_1;\ C = \log_e P_1$

$$f\theta = \log_e P - \log_e P_1. \tag{5}$$

When $\theta = \alpha,\ P = P_2;$

$$f\alpha = \log_e \frac{P_2}{P_1}; \tag{6}$$

$$\frac{P_2}{P_1} = e^{f\alpha}. \tag{7}$$

Example

A belt is in contact with a pulley through an angle of 180 degrees. The tension at one point of tangency is 100 pounds and the coefficient of friction is 0.4. What is the maximum tension at the other point of tangency?

$$\log_e \frac{P_2}{P_1} = 3.1416 \times 0.4 = 1.2566.$$

If a suitable table of Naperian logarithms were available, the ratio of P_2 to P_1 could be found by looking up the number whose logarithm is 1.2566. Without such a table, the Naperian logarithms must be reduced to common logarithms.

$$\log_{10} \frac{P_2}{P_1} = 0.5457,$$

$$\frac{P_2}{P_1} = 3.51;$$

$$P_2 = 100 \times 3.51 = 351 \text{ lb.}$$

The tension of 351 pounds is the greatest value possible without slipping. The actual tension may be very much less than 351 pounds.

Problems

1. A rope is wound twice around a cylindrical post. The coefficient of friction is 0.3. What must be the minimum tension in one end of the rope in order to have a tension of 1,000 pounds in the other end?

Ans. 23.05 lb.

2. A rope is hung over a cylinder whose axis is horizontal. When a load of 20 pounds is hung on one end of the rope, it is found that a load of 60 pounds may be hung on the other end. If the rope were given an additional turn around the cylinder, how many pounds could be hung on one end when there is a load of 20 pounds on the other end? Solve without writing.

3. A cylindrical drum 10 inches in diameter is used to lift a load of 2,000 pounds by means of a 1-inch rope. The drum is turned by means of a pulley 4 feet in diameter which is attached at one end. The pulley is driven by a belt from a second pulley of the same diameter. If the coefficient of friction between the belt and the pulley is 0.4, what must be the tension in the two parts of the belt? *Ans.* 641 lb.; 183 lb.

4. Solve Problem 3 if the second pulley is 1 foot in diameter and its axis is 5 feet from the axis of the drum.

119. Summary.—Friction is a force at the surface of contact of two bodies, which resists the motion of one body along the surface of the other. The force of friction is tangent to the common surface of the two bodies and is opposite the direction which one body is moving or tending to move relative to the other body.

The ratio of the friction to the normal pressure is called the coefficient of friction. The coefficient of moving friction is the value of this ratio when the bodies actually move relative to each other. The coefficient of starting friction is the maximum value of this ratio when there is no motion. The coefficient of starting friction is greater than the coefficient of moving friction.

For solids which are not lubricated, the coefficient of friction is fairly constant with varying pressure and velocity. The coefficient varies greatly with the smoothness of the surface and the kind of material.

Material	Coefficient of Moving Friction
Wood on wood, dry	0.25 to 0.50
Metal on metal, dry	0.15 to 0.20
Leather on metal, dry	0.35 to 0.55

When the surfaces are well lubricated, the coefficient of friction decreases with increase of pressure and is much smaller than for dry surfaces.

The ratio of the friction to the *normal* pressure (the coefficient of friction) is the *tangent* of the angle of friction. The ratio of the friction to the *resultant* pressure is the *sine* of the angle of friction.

If ϕ is the angle of friction, the resultant of all the applied forces, except the normal reaction and the friction, makes an angle ϕ with the direction of the normal.

The coefficient of rolling friction is the distance of the point of application of the resultant reaction at the surface of the rolling body from a line through the center of the rolling body parallel to the resultant of all loads except the force required to roll the body.

The coefficient of rolling friction is practically independent of the radius of the rolling body. The force required to roll a body varies inversely as its diameter.

In the case of a belt running over a pulley,

$$\log_e \frac{P_2}{P_1} = f\alpha,$$

$$\log_{10} \frac{P_2}{P_1} = 0.4343\, f\alpha,$$

in which P_2 is the tension in the belt at one point of tangency, P_1 is the tension at the other point of tangency, f is the coefficient of friction, and α is the angle of contact in radians.

120. Miscellaneous Problems

1. A cylinder, 8 inches in diameter, weighing 6 pounds, is placed on two parallel rods, which make an angle of 15 degrees with the horizontal. A string is wound around the cylinder and supports a weight on the free end. What must be the weight in order to hold the cylinder in equilibrium, and what must be the minimum value of the coefficient of friction? What is the direction of the resultant of the friction and the normal at the points of contact with the rods?

2. A cylinder, 8 inches in diameter, and 24 inches in length, weighs 18 pounds and has its center of gravity at the middle. The cylinder is supported by a rod perpendicular to its length at a distance of 9 inches from one end, and by a cord which is wound around it and supported above. If the coefficient of friction is sufficient to prevent sliding, how far must the cord be placed from the other end of the cylinder? Solve by moments. Also solve graphically.

3. A car is 56 inches wide, center to center of wheels, and 108 inches long, center to center of axles. The center of gravity is 30 inches above the ground and 60 inches in front of the rear axle. If the coefficient of starting friction is 0.30 and the brakes are set so that the wheels will turn, what is the steepest grade which the car can descend without accelerating?

Ans. 7° 01′.

4. What is the steepest grade which it is possible for the car of Problem 3 to ascend? *Ans.* 8° 16′.

CHAPTER XIII

WORK AND MACHINES

121. Work and Energy.—In Art. 39, the work done by a force was defined as the product of the displacement of its point of application multiplied by the component of the force in the direction of the displacement. Expressed algebraically,

$$U = P \cos \alpha \; s, \qquad \text{Formula III}$$

in which P is the force; s is the displacement or amount of motion of its point of application; α is the angle between the direction of the force and the direction of the displacement; and U is the work. Since

$$P \cos \alpha \times s = P \times s \cos \alpha, \qquad (1)$$

work may be defined as the product of the entire force multiplied by the component of the displacement in the direction of the force.

Work is the *scalar* product of two vectors, and is not a vector. Quantities of work may be added as mere numbers with no regard to the directions of the forces and displacements.

Example I

A 10-pound mass is lifted 4 feet vertically upward. Find the work done in lifting it.

The average force is 10 pounds vertically upward. If the body was initially at rest, a force a little greater than 10 pounds was required to set it in motion. As it was brought to rest at the end of the displacement, the force was a little less than 10 pounds. The point of application of the force moves in the direction of the force so that $\cos \alpha = 1$.

$$U = 10 \times 4 = 40 \text{ foot-pounds.}$$

Example II

A 10-pound mass is pulled 12 feet up a 30-degree smooth inclined plane by a force which makes an angle of 15 degrees with the plane. Find the work done.

A resolution parallel to the plane gives,

$$P \cos 15° = 10 \sin 30° = 5 \text{ lb.};$$
$$U = 5 \times 12 = 60 \text{ foot-pounds.}$$

Instead of taking the body up the inclined plane of Fig. 186, it might have been taken horizontally from A to B, and then lifted vertically from

B to *C*. If there is no friction, no work is done in taking the body from *A* to *B* (since the force is normal to the displacement). A force of 10 pounds is required to lift the body vertically upward from *B* to *C*. The displacement from *B* to *C* is 6 feet and the work is 60 foot-pounds.

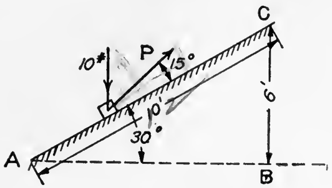


FIG. 186.

When the force varies in magnitude or direction, or when the displacement varies in direction, the total work is calculated from the work of a large number of small displacements. If ds is an element of the displacement so small that the magnitude and direction of the force and the direction of the displacement may be taken as constant throughout that interval, the increment of work is given by the equation,

$$dU = P \cos \alpha \, ds. \tag{2}$$

The total work is the integral of Equation (2),

$$U = \int P \cos \alpha \, ds. \tag{3}$$

Problems

1. A 90-pound mass on a horizontal plane is moved 60 feet north by a force of 50 pounds which is directed north 20 degrees east. Find the work done by this force. *Ans.* 2819 ft.-lb.

2. It requires a force of 12 pounds to stretch a given spring a distance of 1 foot. If the force is proportional to the elongation, what is the work done in stretching the spring 1 foot? Solve by means of the average force and check by integrating Equation 3. *Ans.* 6 ft.-lb.

3. After the spring of Problem 2 has been stretched 1 foot, find the work required to stretch it an additional foot. *Ans.* 18 ft.-lb.

When a body is lifted vertically upward, work is done on it. When the body comes down to its original position, it may do an equal amount of work on another body. When a spring is stretched, work is stored up in it and this work may later be given to another body. Work which is stored up in a body is called *energy*. The energy of a deformed elastic body, or the energy of a body which has been lifted against the force of gravity, is called *potential energy*. Potential energy is sometimes defined as the *energy* of *position*. The energy of a moving body is called *kinetic energy*, or the *energy of motion*. In Example 1, the potential energy of the 10-pound mass, after it has been lifted 4 feet, is 40 foot-pounds more than the energy before it was lifted. This gravitational potential energy may be regarded as the energy of deformed ether. In Problem 2, the potential energy of the spring is 6 foot-pounds when it is stretched 1 foot, and is 24 foot-pounds when it is stretched 2 feet.

Kinetic energy will be discussed in Chapter XVI.

122. Equilibrium by Work.—When a body which is in equilibrium under the action of several forces is displaced, the total work done by all the forces is equal to zero. This is true because, when a body is in equilibrium, the resultant force is zero and the component of the resultant along any direction of displacement is zero. This principle affords a convenient method of solving some problems of equilibrium. The method is especially valuable when the body is constrained to move along some smooth surface. In the application of the method, the body is assumed to suffer a displacement along the surface, and the total work of all the forces is equated to zero. If the direction and magnitude of the forces remain constant through a large displacement, the assumed displacement may be taken as equally large. If either the direction or the magnitude of any force changes with a small displacement, the assumed displacement is taken as an infinitesimal length.

123. Machines.—A body or combination of bodies for the transfer or transformation of work is called a machine. The *input* is the work done on a machine. The *output* is the useful work done by the machine. The ratio of the useful work done by the machine to the input is the *efficiency* of the machine. Suppose that a rope over a single pulley is employed to lift a load of 40 pounds vertically upward, and suppose that the pull on the rope on the other side of the pulley is 50 pounds. When the load is lifted 1 foot, the useful work is 40 foot-pounds. If the rope does not stretch, the input is 50 foot-pounds. The efficiency is 80 per cent. The 10 foot-pounds of work are lost in friction of the pulley, and are transformed into heat. If there were no friction or similar losses, the efficiency would be one hundred per cent.

The force which acts on a machine with a component in the direction of the displacement, and which does positive work on the machine, is called the *effort*. The force which does negative work on the machine, or with which the machine does work, is called the *resistance*. The ratio of the resistance to the effort is called the *mechanical advantage* of the machine. When a force of 20 pounds on one arm of a lever lifts a mass of 60 pounds vertically upward, the mechanical advantage of that lever as a machine is $\frac{60}{20} = 3$. Since the work done by the effort is equal to the negative work of the resistance, (assuming no friction), the mechanical advantage is equal to the ratio of the displacement of the effort to the displacement of the resistance.

124. Inclined Plane.—In Fig. 187, a body on an inclined plane is supported by a force parallel to the plane. The supporting force P and the mechanical advantage will be found by the method of work. If the length of the plane is l, the work done by the force P in pulling the body that distance is Pl. If the height of the plane is h, the negative work of gravity on the body of W pounds mass is Wh. The normal reaction of the plane does no work, since its direction is perpendicular to the displacement. If there is no friction, the input is equal to the output, and

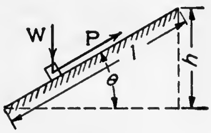


FIG. 187.

$$Pl = Wh, \tag{1}$$

$$P = \frac{Wh}{l} = W \sin \theta \tag{2}$$

Equation (2) is the same as is obtained by a resolution parallel to the inclined plane.

$$\frac{W}{P} = \frac{l}{h}. \tag{3}$$

Equation (3) states that the mechanical advantage of an inclined plane, when the force is applied parallel to the plane, is equal to the ratio of the length of the plane to its height, or is equal to the cosecant of the angle of slope of the plane.

In Fig. 188, the force P, which acts on the body, is horizontal. When the body moves a distance l along the plane, the component of its displacement in the direction of the force is the horizontal projection of that distance. The work done by the force P is Pb, in which b is the horizontal projection of l.

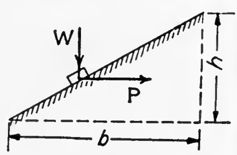


FIG. 188.

$$Pb = Wh, \tag{4}$$

$$P = \frac{Wh}{b} = W \tan \theta. \tag{5}$$

$$\frac{W}{P} = \frac{b}{h} = \text{cotan } \theta. \tag{6}$$

Equation (5) states that the mechanical advantage of an inclined plane, when the force is horizontal, is equal to the ratio of the base of the plane to its height.

When a body is lifted up an inclined plane, the output is in the form of the increased potential energy at the new position.

In any machine, the input is always greater than the output. No work is gained. What is gained is the ability to exert a large force through a small distance by means of a convenient small force acting through a larger distance, or a large displacement with a small force by means of a smaller displacement with a larger force. One may not be able to lift 400 pounds vertically upward a distance of 2 feet, but he may easily accomplish the same result by means of an inclined plane 10 feet in length.

Problems

1. A barrel weighing 240 pounds is lifted a vertical distance of 3 feet by rolling up a plank 12 feet in length as an inclined plane. The force is applied parallel to the plank by means of a frictionless bearing at the axis. If the rolling friction is negligible, what is the work done? What is the force, and what is the mechanical advantage?

2. The barrel of Problem 1 is rolled up the plane by means of a rope which is wound around it several times and then fastened. How far will the end of the rope move while the barrel rolls 12 feet, if the pull is applied parallel to the plane? What is the pull? What is the mechanical advantage? What is the minimum value of the coefficient of friction between the barrel and the plank in order that it may be rolled in this way? (The diameter of the rope is negligible.)

3. A body is pulled up a 30-degree inclined plane by means of a force parallel to the plane. The body is on rollers which have a combined friction equal to a coefficient of sliding friction of 0.02. Find the efficiency.

The useful work is $W\,s.\ \sin 30°$ The normal pressure is $W \cos 30°$. The lost work of friction is $0.02\,W\,s \cos 30°$

$$\text{Useful work} = 0.500\ Ws;$$
$$\text{lost work} = 0.0173\ Ws;$$
$$\text{efficiency} = \frac{0.500}{0.517} = 96.7 \text{ per cent.}$$

Frequently, the body is lifted in a carriage which takes up part of the energy. If the energy of the carriage is not returned in the form of useful work, it represents a loss which must be taken into account in calculating the *commercial* efficiency of the machine.

Problems

4. A mass of 100 pounds is lifted up a 30-degree plane on a carriage which weighs 20 pounds. The pull is by means of a rope parallel to the plane. The coefficient of friction is 0.03. Find the commercial efficiency of the inclined plane and carriage if the carriage, when running back, helps lift another one.

Ans. 93.2 per cent, neglecting the loss in transmission to the second carriage.

5. Solve Problem 4 if the carriage is let down by a brake.

Ans. 79.2 per cent.

6. If the efficiency is 95 per cent, what load can be lifted up a 16-degree inclined plane by a pull of 100 pounds parallel to the plane?

The mechanical advantage is the cosecant of 16 degrees. If the efficiency were 100 per cent,

$$W = 100 \times 3.628 = 362.8 \text{ lb.}$$

Allowing for the losses,

$$W = 362.8 \times 0.95 = 344.7 \text{ lb.}$$

7. If the efficiency is 94 per cent, what force parallel to a 20-degree inclined plane will lift 600 pounds up the plane?

125. Wheel and Axle.—When a wheel of radius r (Fig. 189) rotates through an angle of θ radians, the linear displacement is $r\theta$. If the wheel of radius r is attached to an axle of radius a, the equation of work is

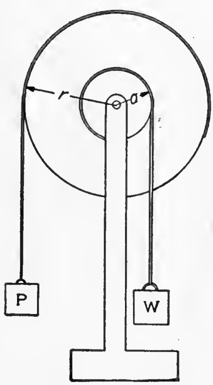


FIG. 189.

$$Wa\theta = Pr\theta; \quad (1)$$

$$\frac{W}{P} = \frac{r}{a}. \quad (2)$$

The mechanical advantage of the wheel and axle is the ratio of the radius (diameter) of the wheel to the radius (diameter) of the axle.

Equation (2) might be derived by moments about the axis. A wheel and axle may be regarded as a continuous lever.

Problems

1. The diameter of a wheel is 2 feet, and the diameter of the axle is 6 inches. The rope on the wheel is ½ inch in diameter and the rope on the axle is 1 inch in diameter. Find the mechanical advantage.

Ans. 3.5.

2. A wheel 40 inches in diameter is attached to an axle 8 inches in diameter. The bearings are 3 inches in diameter and the coefficient of friction is 0.03. The wheel and axle together weigh 120 pounds. Both ropes run vertically downward. Neglecting the diameter of the ropes, find the force required to lift 200 pounds, and find the efficiency.

Ans. 40.8 lb.; 98 per cent.

3. Solve Problem 2 for a load of 100 pounds and for a load of 600 pounds.

4. What force is required with a wheel 4 feet in diameter and an axle 10 inches in diameter to lift a load of 1200 pounds, if the rope on the wheel is 0.5 inch in diameter; the rope on the axle is 1 inch in diameter; and the efficiency is 94 per cent?

5. A body is pulled up a 20-degree inclined plane by means of a rope parallel to the plane. The rope runs from an axle 12 inches in diameter which is turned by a wheel 40 inches in diameter. Neglecting the diameter of the ropes, find the mechanical advantage of the combination. If the combined efficiency is 92 per cent, what force on the wheel will lift 1200 pounds up the plane?

126. Pulleys.—Figure 190 shows a single pulley with a flexible cord. When the effort P moves downward a distance s, the load W moves upward an equal distance. The mechanical advantage is unity.

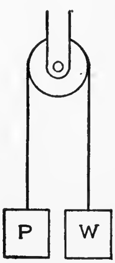


FIG. 190.

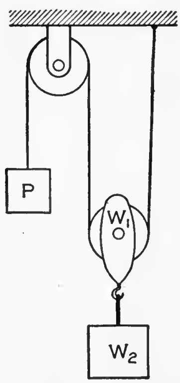


FIG. 191.

In Fig. 191, there is a *fixed* pulley at the top and a single *movable* pulley below. If the movable pulley is lifted a distance s, the rope supporting it is shortened that distance on each side and the part of the rope which passes over the fixed pulley must travel a distance of $2s$.

$$P2s = Ws;$$

$$\frac{W}{P} = 2.$$

The mechanical advantage of a single movable pulley supported by two parts of a continuous rope is 2. This is evident from the fact that the tension in a continuous rope which runs over smooth pulleys is the same in all parts of its length, and two parts of the rope support the load W, while only one part supports the force P.

Since the block and pulley of weight W_1, which form the movable system, must be lifted along with the load W_2, the commercial efficiency is $\frac{W_2}{W_1 + W_2}$ if there are no losses due to friction of the pulleys or bending of the ropes.

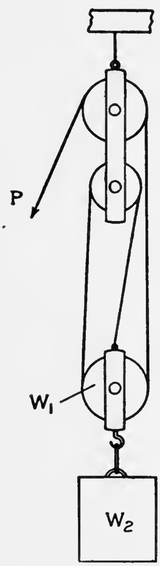


Fig. 192.

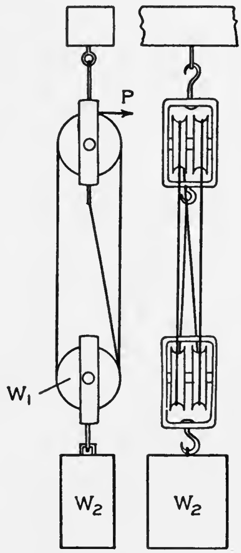


Fig. 193.

In Fig. 192, the movable block carries one pulley and the fixed block carries two pulleys. The rope is attached to the movable block. For clearness, the pulleys in the upper block are made of unequal size and placed one above the other. This, however, is not a customary arrangement. It is evident that the mechanical advantage is 3 when the blocks are so far apart that the three ropes which support the movable block are practically parallel. In the position shown in Fig. 192, the mechanical advantage is less than 3.

Figure 193 shows a system with two pulleys in the movable block. The mechanical advantage is 4 when the blocks are so far apart that the ropes are practically parallel. The pulleys on each block run on the same axis but with different speeds.

In a system of pulleys with one continuous rope, the mechanical advantage is equal to the number of ropes which directly support the weight.

Problems

1. Sketch a system of pulleys arranged as in Fig. 192 with a mechanical advantage of 5.

2. In a system similar to Fig. 193, the mechanical efficiency is 95%. The movable block and pulleys weigh 20 pounds. The weight W_2 is 400 pounds. Find the force required to lift the system, and find the commercial efficiency.

3. A safe weighing 2,400 pounds is pulled up an inclined plane by a block and tackle. The plane is 24 feet long and 5 feet high. The block and tackle has a mechanical advantage of 4 and an efficiency of 95 per cent. The efficiency of the plane is 90 per cent. Find the force and find the required length of rope

4. Sketch a system of pulleys combined with a wheel and axle for a combined mechanical advantage of 12.

127. The Screw.—In lifting a body by means of a screw, the effort travels a distance $2\pi r$ in one revolution, where r is the distance of the effort from the axis. At the same time, the load moves a distance equal to the pitch of the screw. If d is the pitch (distance between threads),

$$Wd = 2\pi Pr;$$

$$\text{mechanical advantage} = \frac{2\pi r}{d}.$$

The efficiency of a screw is small. It must be less than 50 per cent or the screw will run down when the turning moment is released. The screw is a continuous inclined plane with the force applied parallel to the base. If the friction on an inclined plane is less than the component of the load parallel to the plane, the load will slide down. In order to pull a body up a plane where the friction is sufficient to prevent it from sliding down, the force parallel to the plane must be twice as great as the component of the weight. The force parallel to the base must be still greater.

Problem

A screw with 4 threads to the inch is turned by a lever 15 inches long from the axis of the screw to the point of application of the effort. What is the mechanical advantage? If the efficiency is 30 per cent, what is the force required to lift 2,000 pounds? *Ans.* 377,17.7 lb.

128. Differential Appliances.—Figure 194 shows a differentia screw. The part which passes through the nut A is made with one pitch, while the part which passes through the nut B is made with a smaller pitch. If the screw is rotated in a clockwise direction, (as seen by an observer looking toward the head from the right), it will move toward the left through the nut A. If the nut B is held so that it does not rotate with the screw, it will move toward the right relative to the screw, but will move toward the left relative to the nut A. The actual motion of B relative to A during one revolution is equal to the difference of pitch of the two parts of the screw.

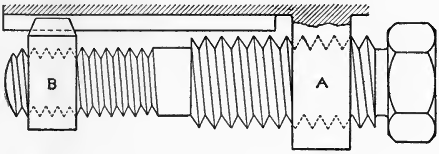


FIG. 194.

Problems

1. The pitch of the right end of the screw of Fig. 194 is three threads to the inch and the pitch of the left end is four threads to the inch. The screw is turned by means of a rod which is 20 inches long from the axis of the screw to the point of application of the force. Find the mechanical advantage. *Ans.* 1508.

2. What would be the mechanical advantage in Problem 1 if the pitches were 8 threads and 9 threads to the inch, respectively?

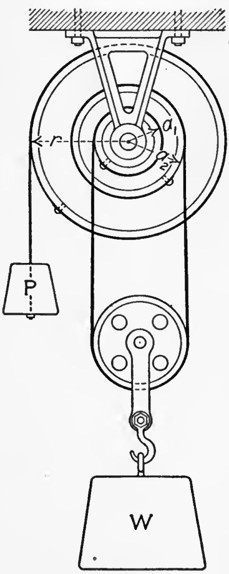


FIG. 195.

Figure 195 shows a differential axle. As part of the rope around the axle of radius a_2 is wound up, the part around the axle of radius a_1 is unwound. In turning through an angle of θ radians, the total vertical rope is decreased in length by an amount $(a_2 - a_1)\theta$ and the load is lifted $\frac{a_2 - a_1}{2}\theta$. The effort moves through a distance $r\theta$, and mechanical advantage $= \frac{2r}{a_2 - a_1}$. The combination is equivalent to an axle of radius $a_2 - a_1$ lifting a single movable pulley with two ropes to support the weight. With this arrangement, the mechanical advantage of a small axle is secured and the axle is sufficiently large to carry the required load.

Problems

3. In Fig. 195, the wheel is 20 inches in diameter. The larger part of the axle is 6 inches in diameter and the smaller part is 5 inches in diameter. Find the mechanical advantage.

4. Find the mechanical advantage of Problem 3 by moments about the axis of the axle.

In many machines it is not easy to determine the mechanical advantage from inspection. The mechanical advantage may always be found by moving the machine and measuring the relative motion of the effort and the resistance. In a differential pulley, for instance, it was found that the chain by which the effort was applied moved 50 inches, while the hook which carried the load moved 1 inch. Evidently the mechanical advantage is 50.

129. The Lever.—The mechanical advantage of the lever is most easily calculated by moments. If the forces are perpendicular to the moment arms, as in Fig. 196, I; or if the lever is straight so that the points of application of all the forces are on the same straight line, and the forces are parallel, the moment equation is

$$Pa = Wb, \tag{1}$$

$$\frac{W}{P} = \frac{a}{b}. \tag{2}$$

Equation (2) states that the mechanical advantage is the ratio of the effort arm to the resistance arm.

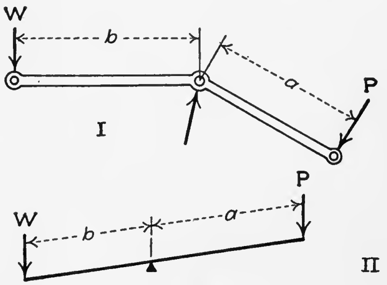


FIG. 196.

If the lever of Fig. 196, I, is turned through a small angle $d\theta$, the work done by the force P is $Pad\theta$, and the negative work of the resistance W is $Wbd\theta$. Equation (1) is obtained by equating these two expressions for work.

In Fig. 196, II, the work of the force P is $Pad\theta \cos \theta$, and the work of the force W is $Wbd\theta \cos \theta$, in which θ is the angle between the lever and a line perpendicular to the forces.

The efficiency of a lever as a machine may be nearly 100 per cent, when the fulcrum is a carefully made knife edge as in a chemical balance.

130. Virtual Work.—The preceding articles of Chapter XIII have dealt with bodies moving with uniform speed. The same conditions apply to a body at rest. If a body in equilibrium be considered as suffering a small displacement, the total work of all the forces will be zero. In Fig. 197, the body of mass **W** which is held on a smooth inclined plane by a force **P** parallel to the plane may be regarded as moved up the plane a distance ds. The body is raised a vertical distance dy, so that

$$- Wdy + Pds = 0 \tag{1}$$

$$P = \frac{Wdy}{ds} = W \sin \theta. \tag{2}$$

Figure 198 shows a bar of length l, mass W, with its center of mass at a distance a from the lower end. The bar rests on a smooth horizontal floor and leans against a smooth vertical wall. The bar is held from slipping by a horizontal force at the bottom. This horizontal force will be computed by the method of work.

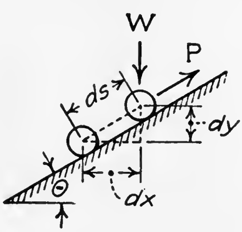


Fig. 197.

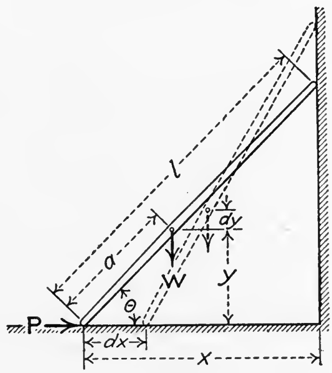


Fig. 198.

When the lower end of the bar is moved a distance dx toward the right, the center of mass is moved upward a distance dy. As the reactions at the wall and the floor are normal to the displacement, their work is zero. This work equation is

$$Pdx = Wdy \tag{3}$$

The relation between dx and dy is found by differentiating an equation between x and y. This relation is conveniently found by means of a third variable. If θ is the angle between the bar and the horizontal, and y is the distance of the center of mass above the floor,

$$y = a \sin \theta, \tag{4}$$

$$dy = a \cos d\theta. \tag{5}$$

If x is the distance of the lower end of the bar from the vertical wall and is positive toward the right,

$$x = -l \cos \theta, \tag{6}$$

$$dx = l \sin \theta \, d\theta. \tag{7}$$

Substitution in Equation (3) gives

$$Pl \sin \theta \, d\theta = Wa \cos \theta \, d\theta, \tag{8}$$

$$\frac{W}{P} = \frac{a \operatorname{cotan} \theta}{l}. \tag{9}$$

This method of determining the conditions of equilibrium by assuming that the body suffers a small displacement and equating the work of all the forces is called the method of *virtual work.*

Problems

1. A ladder 20 feet long, weighing 36 pounds, with its center of gravity 8 feet from the lower end, rests on a smooth horizontal floor and leans against a smooth vertical wall. It is held from slipping by a horizontal force of 16 pounds applied 2 feet from the bottom. Find the position of equilibrium by the method of virtual work. Check by moments and resolutions.

Ans. 45 degrees with the horizontal.

2. The ladder of Problem 1 rests on a smooth horizontal floor and leans against a smooth wall which makes an angle of 15 degrees with the vertical, away from the ladder. It is held from slipping by a horizontal push of 8 pounds at the bottom. Find the position of equilibrium.

Ans. Tan $\theta = 1.5321$; $\theta = 56° \, 52'$ with the horizontal.

3. Find the mechanical advantage of the toggle joint, Fig. 199, in terms of x and the lengths a and b.

$$-P\,dx = W\,dy$$

$$y = \sqrt{a^2 - x^2} + \sqrt{b^2 - x^2},$$

$$dy = \frac{-x\,dx}{\sqrt{a^2 - x^2}} + \frac{-x\,dx}{\sqrt{b^2 - x^2}},$$

$$\frac{W}{P} = \frac{\sqrt{(a^2 - x^2)(b^2 - x^2)}}{xy}$$

4. In Fig. 199, $x = 2$ feet; $a = b = 8$ feet. Find the mechanical advantage.

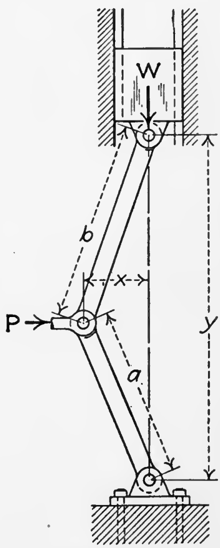


FIG. 199.

Example

Figure 200 shows the crank, connecting rod, and cross-head of an engine. Find the relation of the total piston pressure P to the component of the force perpendicular to the crank. Give the result in terms of the lengths of the crank and the connecting rod and the position of the crank.

$$x = a \cos \theta + b \cos \phi = a \cos \theta + \sqrt{b^2 - a^2 \sin^2\theta};$$
$$-P\,dx = R\,a\,d\theta;$$
$$\frac{R}{P} = \sin \theta + \frac{a \sin \theta \cos \theta}{\sqrt{b^2 - a^2 \sin^2 \theta}} = \sin \theta + \cos \theta \tan \phi.$$

Problems

5. In Fig. 200, find the normal reaction **N** at the cross-head by resolutions, Then find the moment about the axis of the axle at O by a moment equation.

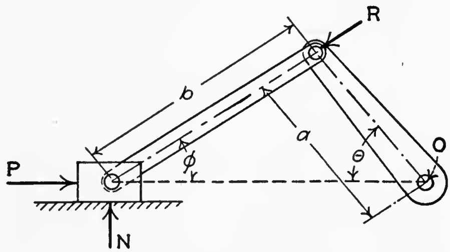


FIG. 200.

regarding the connecting rod and crank as a single body. Compare the result with that obtained by means of the answer of the Example.

6. A bar AB, Fig. 201, of length c is hinged at A and supported by a rope at B. The rope runs over a smooth pulley C at the same level as A, at a distance b therefrom, and carries a load of P pounds on the free end. The bar weighs W pounds and its center of mass is at a distance a from the hinge. Find the ratio of W to P at any position.

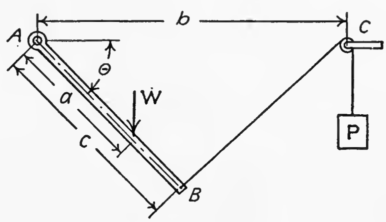


FIG. 201.

Let $BC = s$, and let the vertical distance of the center of mass below the hinge be represented by y, which is taken as positive downward.

$$W dy = P\,ds;$$
$$\frac{W}{P} = \frac{bc \sin \theta}{a \cos \theta \sqrt{b^2 + c^2 - 2bc \cos \theta}}.$$

7. In Problem 6, let $W = 12$ pounds, $P = 10$ pounds, $a = 4$ feet, $b = 8$ feet, and $c = 6$ feet. Find the equation for the angle θ.

Ans. $0.96 \cos^3 \theta - 2 \cos^2 \theta + 1 = 0.$

8. Solve Problem 7 for *cos* θ by the method of trial and error.

$\cos\theta$	$0.96\cos^3\theta$	$-2\cos^2\theta$	$f(\theta)$
1	0.96	−2	−0.04
0	0.	0	+1

The function of θ changes sign between $\cos\theta = 0$ and $\cos\theta = 1$. There must be one real root or three real roots between these values. One root is evidently much closer to unity than to zero. Beginning with $\cos\theta = 0.9$,

0.9	0.69984	−1.62	+0.07984
0.95	0.82308	−1.805	+0.01808
0.96	0.84936	−1.84320	+0.00616
0.97	0.87616	−1.88180	−0.00564
0.965	0.862687	−1.862450	+0.000237
0.966	0.865372	−1.866312	−0.000940

An interpolation of the last two results gives $\cos\theta = 0.9652$, $\theta = 15° 10'$.

The remaining roots may be found by a similar process. It is better, however, to divide the cubic equation by $\cos\theta - 0.9652$. The quotient is an equation of the second degree, which may be solved by the ordinary methods for quadratics. One of the roots of this quadratic is a solution of the problem of mechanics. The other root is a solution of the mathematical equation, but is geometrically impossible. Draw the space diagram for the possible result.

9. A straight bar of length l and mass W, with its center of mass at a distance a from the lower end, rests on a smooth horizontal floor and leans over the edge of a smooth wall. The vertical height from the floor to the point of contact with the wall is h. A rope attached to the bottom of the bar runs horizontally over a smooth pulley and supports a mass of P pounds. Find the equation for the position of equilibrium by the method of virtual work. *Ans.* $Wa(\cos\theta - \cos^3\theta) = Ph$.

10. Solve Problem 9 by trial and error for the following cases:

	a	h	W	P
20′	10′	12′	15#	4#
16′	10′	12′	30#	8#
20′	10′	12′	10#	5#

In many problems which may be solved by virtual work, there is only one term in the work equation. For equilibrium, this term must equal zero. If the force in this term is the weight of some body, the work is zero when the point of application of this force is moving in a horizontal direction. The path of the point of application is a curve of some form. If the curve is concave upward at the point at which the tangent is horizontal, the equilibrium is stable. If it is concave downward, the equilibrium is unstable.

Problems

11. A rod of length l, Fig. 202, with its center of mass at a distance a from one end, slides inside a smooth hemispherical bowl of radius r. Find the position of equilibrium if l is less than $2r$.

It is evident that the path of the center of mass is a circle with its center at the center of the sphere. Equilibrium exists when the center of mass is at its lowest point directly under the center of the sphere. In this position, find the angle which the bar makes with the horizontal.

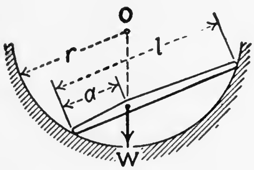


Fig. 202.

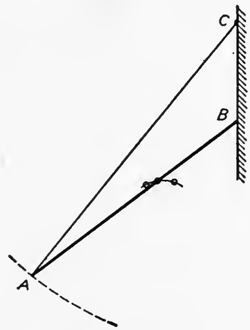


Fig. 203.

12. Solve Problem 11 when $l = 6$ inches, $a = 2$ inches, and $r = 4$ inches.

Ans. Bar makes an angle of 20° 42′ with the horizontal.

13. A bar AB, Fig. 203, is 8 feet long and has its center of mass 5 feet from A. The end B rests against a smooth vertical wall. The end A is supported by a cord, 10 feet in length, which is fastened to the vertical wall directly over the end B. Find the position of equilibrium. Solve by virtual work and check by the direction condition of equilibrium.

Ans. The bar makes an angle of 53° 05′ with the vertical.

131. Character of Equilibrium.—It is often easy to determine the kind of equilibrium by means of the methods of work. In Problem 11 of the preceding article, the path of the center of mass is a circle and equilibrium occurs at the lowest point. Work must be done on the bar to lift its center of mass in either direction up the curve. If the center of mass is displaced up the curve, the force of gravity will do work on it in bringing it back to the position of equilibrium. The equilibrium is stable.

If the bar of Problem 11 were inside a complete sphere, and if the ends of the bar passed through slots in the surface of the sphere and were bent over so that the bar could slide inside the top, another point of equilibrium could be found at the top of the sphere. The path of the center of mass would be concave downward at this position. If the bar were displaced slightly from this position, the force of gravity would tend to displace it still farther, and it would not return. In this case, the equilibrium is unstable. In many cases, when a body is displaced from its position of unstable equilibrium, it will move to the position of stable equilibrium.

A position at which the total work of a small displacement is zero, is a position at which the potential energy of the system is a maximum or a minimum. If gravity is the only force which does work on a body, the potential energy is a maximum when the path is concave downwards, and a minimum when the path is concave upwards.

In Fig. 203, the location of the center of mass for one position on each side of the position of equilibrium is represented by a small circle. It is evident that the locus of the center of mass is concave downward, and that the equilibrium is unstable.

Figure 201 applies to Problem 6 of Art. 130 for one position of equilibrium. At the other position of equilibrium, the bar is above the hinge and toward the left. In this second position, the center of mass of the bar is higher than that of Fig. 201. The length BC is also greater; hence the load P is higher. Consequently the potential energy in the second position is greater than in Fig. 201. Figure 201 shows the position of stable equilibrium. At the other position, the equilibrium is unstable.

132. The Beam Balance.—Figure 204 shows the fundamental parts of a beam balance with equal arms. In an accurately

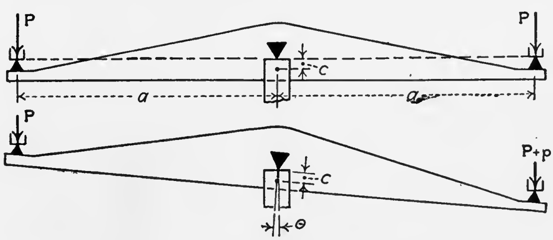


FIG. 204.

constructed balance, the three knife edges are in the same plane and the center of gravity of the beam is located a little below the central knife edge. When equal loads are applied to the end knife edges, the resultant of these loads falls on the central knife edge and does no work when the beam is deflected. When a small additional load p is placed on the right knife edge, the beam is turned through an angle θ in a clockwise direction. If W is the weight of the beam, c is the distance of its center of gravity below the central knife edge, and a is the distance between knife edges, a moment equation gives

$$Wc \sin \theta = pa \cos \theta; \tag{1}$$

$$\tan \theta = \frac{pa}{Wc}. \tag{2}$$

For small angles, the tangent is equal to the arc in radians and equation (2) becomes,

$$\theta = \frac{pa}{Wc}. \tag{3}$$

Equations (2) and (3) give the *sensibility* of the balance. The sensibility of a balance is the angle through which the beam is turned by unit additional load on one knife edge. Delicate balances are provided with a pointer which projects downward and moves in front of a graduated scale. The sensibility of the balance is usually expressed in terms of the scale reading.

Problems

1. A balance beam weighs 50 grams and its center of gravity is 0.008 inch below the central knife edge. The knife edges are 4 inches apart and are in the same plane. The pointer is 10 inches long. How much will the end of the pointer move when 1 milligram is placed on one pan?

Ans. 0.10 inch.

2. The center of gravity of a balance beam is frequently adjusted by means of a nut, which may be turned up or down on a screw at the top of the beam. (This screw is not shown in Fig. 204.) In Problem 1, the beam weighs 48 grams and the nut weighs 2 grams. How much must the nut be moved to double the sensibility?

If the central knife edge is above the plane of the end knife edges, the resultant of equal loads on the pans is a load of $2P$ units directly below the central knife edge. In order to deflect the beam, work must be done on this load. The sensibility of such a balance is diminished as the load is increased. If d is the distance of the central knife edge above the plane of the end knife edges, the moment equation becomes

$$pa \cos \theta = Wc \sin \theta + 2Pd \sin \theta; \tag{4}$$

$$\tan \theta = \frac{pa}{Wc + 2Pd}. \tag{5}$$

When the central knife edge is above the plane of the end knife edges, the equilibrium is stable for all loads.

Problem

3. In the beam of Problem 1, the central knife edge is 0.0004 inch above the plane of the end knife edges. When the load on each pan is 50 grams, find the deflection of the pointer for 1 milligram additional load on one pan.

Ans. 0.091 inch.

When the central knife edge is below the plane of the end knife edges a distance d, the moment of the resultant force $2P$ changes sign and Equation (5) becomes

$$\tan \theta = \frac{pa}{Wc - 2Pd}. \tag{6}$$

The sensibility is increased with increased load. If the load is too great, the second term of the denominator becomes larger than the first term and the equilibrium is unstable.

When a balance is loaded, the beam is slightly bent. The central knife edge may be a little below the plane of the end knife edges when the total load is small and may come above their plane when the total load is larger. In a beam of this kind, the sensibility may increase with a small increase of total load and decrease with a larger load.

The theory above applies to balances with equal arms. The same principles obtain when the arms are not equal. The only difference is that the sum of the loads on the two pans replaces the load $2P$ of the formulas.

The arms of a balance, which are supposed to be equal, may not be equal. Let a be the length of one arm and b the length of the other. Let P be the load on the arm of length a which balances an unknown load W on the other arm.

$$Pa = Wb. \tag{7}$$

Now put the load W on the arm of length a and suppose that it is balanced by a load Q on the other arm.

$$Qb = Wa, \tag{8}$$

$$W = \sqrt{PQ}. \tag{9}$$

The true weight is the geometric mean of the apparent weights. The arithmetic mean of the apparent weights is sufficiently exact for all ordinary purposes.

If W is eliminated from Equations (7) and (8),

$$\frac{a^2}{b^2} = \frac{P}{Q}; \tag{10}$$

$$\frac{a}{b} = \sqrt{\frac{P}{Q}} = \left(1 + \frac{P - Q}{Q}\right)^{\frac{1}{2}} = 1 + \frac{P - Q}{2Q} \text{ approximately.}$$

Problems

4. An unknown load on the left pan of a balance is balanced by 100.012 grams on the right pan. When the unknown load is placed on the right pan, it is balanced by 99.976 grams on the left pan. Find the true weight of the body. What is the difference between the geometric mean of Equation (9) and the arithmetic mean?

5. Find the ratio of the arms in Problem 4.

133. Surface of Equilibrium.—It is sometimes desirable to find a surface of such form that a body under a given set of forces will be in equilibrium at any point on it. The reaction of the surface at every point must be opposite the direction of the resultant of the applied forces. This means that the resultant of the applied forces must everywhere be normal to the surface. Such a surface is called an *equipotential surface* for the forces involved. The surface of a still lake is an equipotential surface for the force of gravity.

Since the resultant force is normal to an equipotential surface, no work is done in moving a body from one part of the surface to another. If H_x is the component in the direction of the X axis of the resultant of all the forces at the surface (except the reaction of the surface), the work done in displacing a body a distance dx is $H_x dx$. If V is the component in the direction of the Y axis, and H_z is the component in the direction of the Z axis, the total work of a displacement in the surface is

$$U = H_x dx + V dy + H_z dz = 0. \tag{1}$$

Example

The centrifugal force of a rotating body varies as the distance from the axis of rotation. A body of liquid rotates at constant speed about a vertical axis. Find the equation of the intersection of the surface of the liquid with the XY plane.

The horizontal force on a unit of mass is kx and the vertical force is -1. Equation (1) becomes,

$$kxdx - 1dy = 0,$$

$$dy = kxdx.$$

$$y = \frac{kx^2}{2} + C.$$

The curve is a parabola with the axis vertical.

Problems

1. A mass of 10 pounds slides on a smooth wire in a vertical plane. The mass is attached to a cord which passes over a pulley in the same vertical plane. The other end of the cord is attached to a spring. A force of 1 pound stretches this spring 1 inch. When there is no elongation of the spring, the end of the cord just reaches the pulley. If the wire passes through a point 12 inches above the pulley, find its equation in order that the mass may be in equilibrium at any point on it.

The tension in the cord is equal to $-r$ where r is the distance of the end of the cord from the pulley. The horizontal component of this tension is the total tension multiplied by the cosine of the angle which the cord makes with the horizontal. This cosine is $\frac{x}{r}$.

$$H_x = -x; \quad V = -y - 10;$$
$$xdx + (y + 10)dy = 0;$$
$$x^2 + y^2 + 20y = C.$$

When $x = 0$, $y = 12$; $C = 384$;

$$x^2 + (y + 10)^2 = 22^2.$$

The wire is in the form of a circle of 22 inches radius with its center 10 inches below the pulley.

2. Solve Problem 1 if the end of the cord is 4 inches from the pulley when the tension in it is zero. (The pull $= -(r - 4)$.)

Ans. $x^2 + y^2 + 20y - 4\sqrt{x^2 + y^2} = 336.$

3. A body slides on a curved wire in a horizontal plane. It is fastened to two cords. One cord passes over a pulley at A and is attached to a spring which requires a pull of k_1 pounds to stretch it unit distance. The second cord passes over a pulley at B and is attached to a spring which requires a pull of k_2 pounds to stretch it unit distance. The points A and B are at a distance c apart in the direction of the X axis. Find the equation of the wire if the body is in equilibrium at any point on it.

Ans. $(k_1 + k_2)(x^2 + y^2) - 2k_2cx = C$

in which the origin of coordinates is located at A, and the point B lies on the X axis to the right of A.

134. Summary.—The amount of work done by a force is the product of the magnitude of the force multiplied by the component of the displacement of its point of application in the direction of the force

$$U = P \times s \cos \alpha = P \cos\alpha \times s. \qquad \text{Formula III}$$

The second form of the formula states that work is the product of the displacement multiplied by the component of the force in the direction of the displacement.

When the direction or magnitude of the force varies, or when the direction of the displacement varies, the increment of work is given by

$$dU = P \cos \alpha \, ds.$$

A machine transmits energy. The ratio of the *work* done by a machine to the *work* done on it is called the *efficiency* of the machine. The ratio of the *force* exerted by a machine to the *force* applied to it is called the *mechanical advantage* of the machine. The *efficiency* of a machine is never quite equal to unity. The mechanical advantage may have any value.

Problems of equilibrium may be solved by means of work equations. In the method of virtual work, the body or system is assumed to suffer a small displacement and the work of this displacement is equated to zero.

Equilibrium may be stable, unstable, or neutral. The potential energy of a body in stable equilibrium is a minimum. Work must be done on the body to move it from the position of equilibrium. The potential energy of a body in unstable equilibrium is a maximum. If the body is slightly displaced from the position of equilibrium, work must be done on it to move it back to that position. When a body is in neutral equilibrium, no work is required to move it in either direction.

When the force is applied parallel to an inclined plane, the mechanical advantage is the ratio of the length of the plane to its height. When the force is applied parallel to the base of an inclined plane, the mechanical advantage is the ratio of the base to the height.

The mechanical advantage of a wheel and axle is the ratio of the diameter of the wheel to the diameter of the axle.

When a continuous rope is used, the mechanical advantage of a system of pulleys is equal to the number of parts of the rope which support the weight.

The mechanical advantage of a screw is the ratio of the circumference of the circle in which the effort moves to the pitch of the screw. The efficiency of a screw is less than 50 per cent.

The mechanical advantage of a lever is the ratio of the length of the effort arm to the length of the resistance arm. The efficiency of a lever may be very high.

The sensibility of a balance is proportional to the length of the arms and inversely proportional to the mass of the beam and the distance of its center of gravity below the central knife edge.

If the central knife edge is above the plane of the end knife edges, the sensibility is decreased with increased load. If the central knife edge is below the plane of the end knife edges, the sensibility is increased with increased load. If there is much difference in this direction, the equilibrium may be unstable with a large load.

If the arms of a balance are unequal, the load must be weighed first on one pan and then on the other. The true weight is the geometric mean of the two weighings. The arithmetic mean is sufficiently exact in a fairly good balance.

On an equipotential surface

$$H_x dx + V dy + H_z dz = 0.$$

CHAPTER XIV

MOMENT OF INERTIA OF SOLIDS

135. Definition.—The moment of inertia of a solid with respect to an axis may be defined *mathematically* as the sum of the products obtained by multiplying each element of mass by the square of its distance from the axis. Expressed algebraically,

$$I = \Sigma mr^2, \tag{1}$$

in which I is the moment of inertia, m is an element of mass, and r is the distance of the element from the axis. The element must be of such form that all parts are at the same distance from the axis. If the elements are infinitesimal, Equation (1) is written

$$I = \int r^2 dm, \qquad \text{Formula XI}$$

in which dm is the element of mass.

Figure 205 shows the form of the element required to find the moment of inertia with respect to the Z axis OZ. The element BB' is parallel to the axis OZ. If its transverse dimensions are infinitesimal, all parts of the element are at the same distance from OZ, within infinitesimal limits. The element may also be infinitesimal parallel to the axis or it may extend entirely through the body in this direction as shown in Fig. 205. The cross-section of the element in the plane perpendicular to the axis may be rectangular as shown, or may have the form necessary for integration with polar coördinates. The element may be a hollow cylinder of radius r and thickness dr, which extends through the body parallel to the axis.

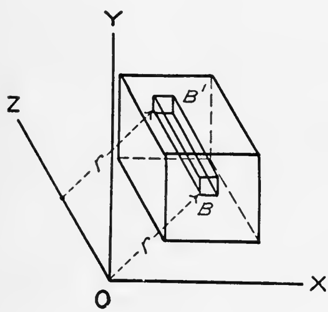


FIG. 205.

136. Moment of Inertia by Integration.—Figure 206 shows one end of a cylinder of radius a. It is desired to find the moment of inertia of this cylinder with respect to its axis, which is perpendicular to the plane of the paper.

Polar coördinates are best suited for this problem. The element of volume parallel to the axis of the cylinder has the cross-section $dA = rd\theta dr$. If l is the length of the cylinder, the element of volume is

$$dV = lrd\theta dr. \tag{1}$$

If ρ is the density, or mass per unit volume,

$$dm = \rho lrd\theta dr. \tag{2}$$

$$I = \int r^2 dm = \rho l \iint r^3 d\theta dr. \tag{3}$$

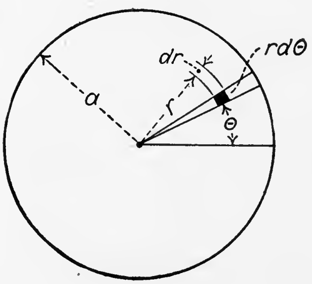


FIG. 206.

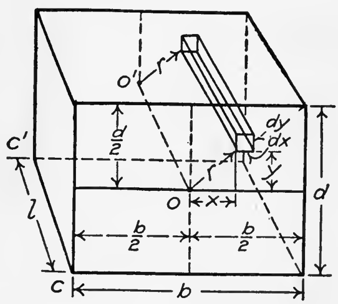


FIG. 207.

Integrating first with respect to r and putting in the limits $r = 0$ and $r = a$,

$$I = \frac{\rho l a^4}{4} d\theta. \tag{4}$$

Integrating next with respect to θ and putting in the limits $\theta = 0$ and $\theta = 2\pi$,

$$I = \frac{\rho l a^4}{4} \Big[\theta\Big]_0^{2\pi} = \frac{\rho l \pi a^4}{2}. \tag{5}$$

The mass of the cylinder is

$$m = \rho l \pi a^2,$$

which substituted in Equation (5) gives

$$I = \frac{ma^2}{2}. \qquad \text{Formula XII}$$

Figure 207 represents a rectangular parallelopiped of length l, breadth b, and depth d. It is desired to find its moment of inertia with respect to the axis of symmetry parallel to its length.

The line OO' is the axis. The element of volume is $dV = ldxdy$.

The element of mass is $dm = \rho l dx dy$. With O–O' as the axis of coördinates,

$$r^2 = x^2 + y^2;$$

$$I = \int r^2 dm = \rho l \iint (x^2 + y^2) dx dy \tag{6}$$

The limits for x are $-\frac{b}{2}$ and $+\frac{b}{2}$. The limits for y are $-\frac{d}{2}$ and $+\frac{d}{2}$.

$$I = \rho l \left(\frac{b^3 d}{12} + \frac{bd^3}{12}\right) = \rho l\, bd \frac{b^2 + d^2}{12} = m \frac{b^2 + d^2}{12}.$$

Problems

1. Find the moment of inertia of the rectangular parallelopiped of Fig. 207 with respect to the axis CC'.

$$Ans.\ I = m \frac{b^2 + d^2}{3}.$$

2. Figure 208 shows a triangular prism of length l. The cross-section is a right-angled triangle. Find the moment of inertia with respect to the edge OO'.

$$Ans.\quad I = \rho l b d \frac{b^2 + d^2}{12} = m \frac{b^2 + d^2}{6}.$$

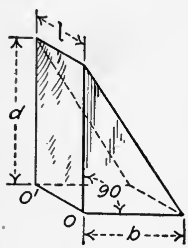


FIG. 208.

3. Find the moment of inertia of a right circular cone of height h and diameter $2a$ with respect to its axis.

Build up the cone of a series of flat disks parallel to the base as shown in Fig. 209. The radius of each disk is r and its thickness is dy. The distance y may be measured from the base, as in Fig. 209 or from the vertex. The expression for the moment of inertia of the disk is written by means of Formula XII. The radius r is expressed in terms of y and the constants; the moment of inertia is found by a single integration.

$$Ans.\quad I = \frac{\rho \pi h a^4}{10} = \frac{3ma^2}{10}.$$

4. Solve Problem 3 by building up the cone of hollow cylinders of radius r and thickness dr, coaxial with the cone. (Fig. 210.)

The solutions of Problems 3 and 4 are instances of the integration of an element of volume $r d\theta dr dy$. Integrating θ first gives a hollow ring. Integrating y next builds the rings into a hollow cylinder. Integrating r last expands these cylinders to form the cone. For Fig. 209, r may be integrated first and θ next, or θ may be integrated first and r next. In either case, the result is the thin disk. The final integration with respect to y builds up the cone of a series of such disks.

5. Find the moment of inertia of a sphere with respect to a diameter, using disks as elements.

$$Ans.\quad I = \frac{8\pi\rho a^5}{15} = \frac{2ma^2}{5}.$$

6. Solve Problem 5 by a single integration, using hollow cylinders as elements of volume.

7. By means of Equation (5) Problem 5, find the moment of inertia of a hollow cylinder of inside radius b and outside radius a.

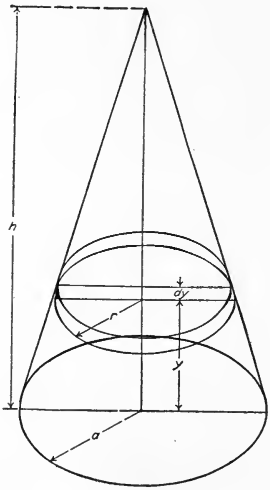


FIG. 209.

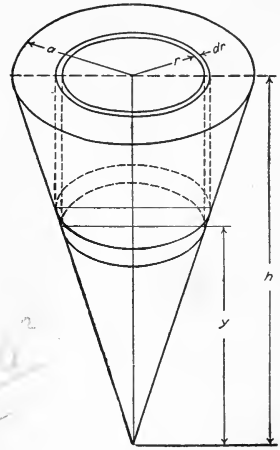


FIG. 210.

8. A right circular cone, 10 inches in diameter, and 12 inches high, has a coaxial cylindrical hole, 4 inches in diameter, cut through it. By means of the results of the preceding examples and problems, find the moment of inertia of the remainder. *Ans.* $I = 615.6\ \pi\rho = 1934\rho$.

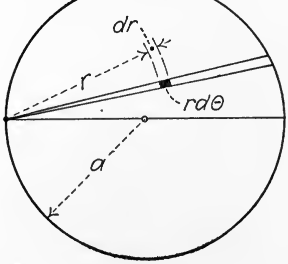


FIG. 211.

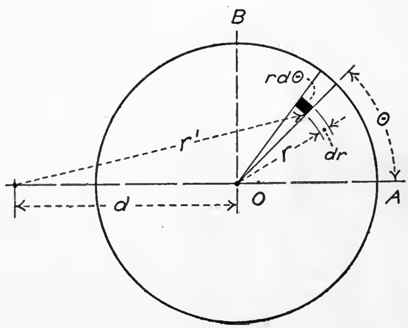


FIG. 212.

9. Find the moment of inertia of a homogeneous solid cylinder of length l and radius a with respect to a line in its surface parallel to its axis. The end of this cylinder is shown in Fig. 211.

10. Find the moment of inertia of a homogeneous solid cylinder with respect to an axis parallel to the axis of the cylinder at a distance d therefrom. Figure 212 shows the end of the cylinder. The axis of inertia is perpendicular to the plane of the paper. For simplicity, the origin of coördinates is taken at the center of the circle. The distance r of Formula XI is r' of the figure. This distance must be expressed in terms of r, θ, and d.

Ans. $I = m\left(\frac{a^2}{2} + d^2\right)$.

11. In Problem 10, find the moment of inertia of the first quadrant alone.

12. Find the moment of inertia with respect to the Z axis of the tetrahedron bounded by the three coordinate planes, and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Ans. $I = \frac{\rho abc}{60}(a^2 + b^2) = \frac{m}{10}(a^2 + b^2)$.

137. Radius of Gyration.—The radius of gyration of a body is the distance from the axis at which all the mass of the body must be placed in order that the moment of inertia may be the same as that of the actual body. It is defined mathematically by the equation

$$k^2 = \frac{I}{m}, \qquad \text{Formula XIII}$$

in which k is the radius of gyration.

The moment of inertia of a homogeneous solid cylinder with respect to its axis is $\frac{ma^2}{2}$; and $k^2 = \frac{a^2}{2}$. If a homogeneous solid cylinder of given mass is replaced by a thin hollow cylindrical shell of the same mass, the moment of inertia of the shell will equal that of the cylinder, provided the radius of the shell is 0.7071 of the radius of the cylinder.

Problems

1. Find the radius of gyration of a homogeneous solid sphere 20 inches in diameter.

Ans. $k = \sqrt{40} = 6.32$ in.

2. Find the radius of gyration of a homogeneous hollow cylinder, of which the outside diameter is 16 inches and the inside diameter is 10 inches.

Ans. $k = 6.67$ in.

3. Find the radius of gyration of a homogeneous hollow cylinder, of which the outside diameter is 16 inches and the inside diameter is 15 inches.

Ans. $k = 7.75$ in.

4. Derive the expression for the square of the radius of gyration of a hollow cylinder, of which the outside radius is a and the inside radius is b.

Ans. $k^2 = \frac{a^2 + b^2}{2}$.

5. Derive the expression for the square of the radius of gyration of a homogeneous solid cone of radius a with respect to the axis of the cone.

Ans. $k^2 = \frac{3a^2}{10}$.

138. Transfer of Axis.—The calculation of the moment of inertia about any axis is frequently a laborious process. The work may often be simplified by means of a transfer of the axis. If the moment of inertia of a body is known with respect to some axis, its moment of inertia with respect to any other axis parallel to that axis may be found by simple algebraic methods.

In Fig. 213, the moment of inertia with respect to the line OO' is desired. Let BB' represent any element of the body parallel to OO'. The distance from OO' to BB' is r.

$$I = \int r^2 dm. \tag{1}$$

Suppose that the moment of inertia of the body is known with respect to the axis CC', and suppose that CC' is parallel to OO'

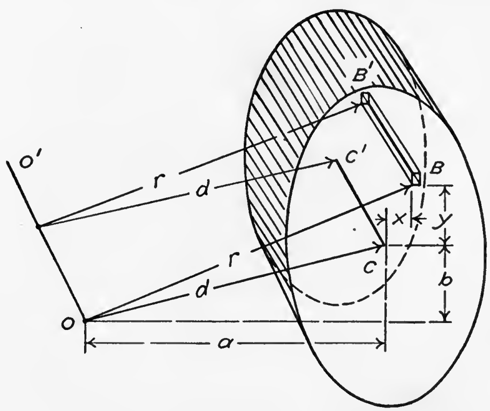


FIG. 213.

at a distance d. The horizontal component of d is a and the vertical component is b.

$$d^2 = a^2 + b^2. \tag{2}$$

The coördinates of the element BB' with respect to CC' as the Z axis are x and y.

$$r^2 = (a + x)^2 + (b + y)^2. \tag{3}$$

$$I = \int (a^2 + 2ax + x^2 + b^2 + 2by + y^2)dm; \tag{4}$$

$$I = \int (a^2 + b^2)dm + \int (x^2 + y^2)dm + 2a\int xdm + 2b\int ydm. \tag{5}$$

The first term of the second member of Equation (5) is

$$(a^2 + b^2)\int dm = md^2.$$

The second term is the moment of inertia with respect to CC'. This moment of inertia may be represented by I_0.

$$\int(x^2 + y^2)dm = I_0.$$

The third term is $2a\int xdm$. The expression xdm is the moment of the element with respect to the vertical plane through CC'. The expression $\int xdm$ is the moment of the entire volume with respect to this plane. If the vertical plane through CC' passes through the center of gravity of the body, the moment is zero. Similarly, the last term is zero if the horizontal plane through CC' passes through the center of gravity of the body. If the *line* CC' passes through the center of gravity of the body, the last two terms of Equation (5) vanish and

$$I = I_0 + md^2, \qquad \text{Formula XIV}$$

in which I_0 is the moment of inertia with respect to an axis through the center of gravity, I is the moment of inertia with respect to a parallel axis at a distance d from the center of gravity, and m is the mass of the body.

Formula XIV affords a convenient method of finding the moment of inertia of a body with respect to any axis, if the moment of inertia is known with respect to a parallel axis through the center of gravity. If the moment of inertia is known for some axis which does *not* pass through the center of gravity, and if it is desired for some other axis which does *not* pass through the center of gravity, Formula XIV may be used to transfer *from* the first axis *to* the center of gravity and then to transfer *from* the center of gravity *to* the second axis.

If k_0 is the radius of gyration with respect to an axis through the center of gravity,

$$I_0 = mk_0^2.$$

If k is the radius of gyration with respect to the axis OO',

$$k^2 = \frac{I}{m} = \frac{mk_0^2 + md^2}{m} = k_0^2 + d^2. \qquad \text{Formula XV}$$

Problems

1. Solve Problem 9 of Art. 136 by means of Formula XIV.

2. Solve Problem 10 of Art. 136 by means of Formula XIV.

3. A solid sphere is 12 inches in diameter. Find its radius of gyration with respect to an axis 10 inches from its center. *Ans.* $k = 10.70$ in.

4. Find the radius of gyration of a hollow cylinder, 8 inches outside diameter and 5 inches inside diameter, with respect to an axis 10 inches from the axis of the cylinder.

5. A circular disk, 20 inches in diameter, has 4 holes drilled through it. The diameter of each hole is 5 inches and the center is 4 inches from the center of the disk. Find the radius of gyration. *Ans.* $k = 7.76$ inches.

6. Knowing the moment of inertia of a sphere and the location of the center of gravity of a hemisphere, find the moment of inertia of a hemisphere with respect to a tangent in a plane parallel to the plane which bounds the hemisphere. Solve by means of Formula XIV. Solve also by means of Equation (5).

139. Moment of Inertia of a Thin Plate.—The moment of inertia of a retangular parallelopiped of length l, breadth b, and thickness d, with respect to one edge parallel to its length (Fig. 214) is

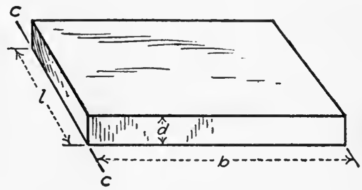


FIG. 214.

$$I = \rho lbd\,\frac{b^2 + d^2}{3} = m\,\frac{b^2 + d^2}{3}.$$

(See Problem 1 of Art. 136). If the thickness is small compared with the breadth, its square may be relatively negligible and may, therefore, be neglected. For instance, if $b = 100$ inches and $d = 1$ inch, the error made by dropping d^2 is only one part in ten thousand. The moment of inertia of a long, slender rod with respect to an axis through one end perpendicular to its length is $m\,\frac{l^2}{3}$, and the moment of inertia with respect to an axis through the middle perpendicular to its length is $m\,\frac{l^2}{12}$. These formulas are derived from the results for a parallelopiped by neglecting the thickness and interchanging l and b.

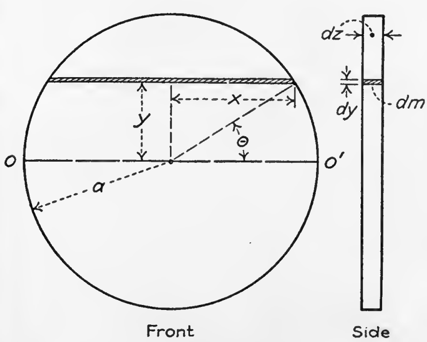


FIG. 215.

Problems

1. Find the moment of inertia of a thin circular disk of thickness dz with respect to a diameter.

If the horizontal diameter, OO' Fig. 215, is taken as the axis of reference, the element of volume parallel to this diameter is $2xdydz$.

The thickness dz is so small (many times less than shown in the side view of Fig. 215) that its square may be neglected in calculating r.

$$r = y,$$
$$I = \rho dz \int 2xy^2dy,$$

which is best solved by substituting $x = a \cos \theta$, $y = a \sin \theta$.

$$I = 2\rho dza^4 \int \cos^2 \theta \sin^2 \theta d\theta = \rho a^4 dz \int \frac{\sin^2 2\theta}{2} d\theta;$$

$$I = \rho a^4 dz \int \frac{1 - \cos 4\theta}{4} d\theta = \rho a^4 dz \left[\frac{\theta}{4} - \frac{\sin 4\theta}{16}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}},$$

$$I = \rho dz \frac{\pi a^4}{4} = \frac{ma^2}{4}.$$

This moment of inertia is one half that of the disk with respect to its axis.

2. Find the moment of inertia of a rectangular plate of breadth b, length l, and thickness dz, with respect to an axis through its center parallel to its length. *Ans.* $I = \rho ldz \frac{b^3}{12} = \frac{mb^2}{12}$.

3. Find the moment of inertia of a triangular plate of height h, base b, and thickness dz, with respect to an axis through its center of gravity parallel to its base.

Find the moment of inertia first with respect to an axis through the vertex parallel to the base. Transfer to a parallel axis through the center of gravity by means of Formula XIV.

Ans. $I = \rho bdz \frac{h^3}{36} = \frac{mh^2}{18}$.

140. Plate Elements.—It is frequently convenient to build up a solid of a series of thin plates. The moment of inertia of each plate is found for an axis through its center of gravity and then transferred to the required axis. Finally, the moment of inertia of the entire solid is found by integration.

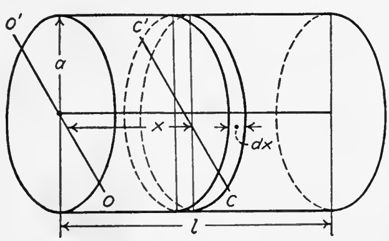


FIG. 216.

Example

Find the moment of inertia of a homogeneous solid cylinder of length l, and radius a with respect to a diameter at one end. (Fig. 216.)

From Problem 1 of Art. 138, the moment of inertia of a disk of radius a and thickness dx with respect to the diameter CC' is $\rho\frac{\pi a^4}{4}dx$. The moment of inertia with respect to the parallel axis OO' at a distance x from CC' is

$$I = I_0 + mx^2 = \rho\pi a^2x^2dx + \rho\pi\frac{a^4}{4}dx; \quad (1)$$

$$I = \rho\pi\left[\frac{a^2x^3}{3} + \frac{a^4x}{4}\right]_0^l = \rho\pi l a^2\left(\frac{l^2}{3} + \frac{a^2}{4}\right); \tag{2}$$

$$I = m\left(\frac{l^2}{3} + \frac{a^2}{4}\right). \tag{3}$$

Problems

1. Find the moment of inertia of a homogeneous solid cylinder with respect to a diameter at the middle. *Ans.* $I = m\left(\frac{l^2}{12} + \frac{a^2}{4}\right)$.

2. Find the moment of inertia of a circular rod with respect to an axis through one end perpendicular to its length, if the diameter is so small that its square is negligible compared with the square of the length.

Ans. $I = \frac{ml^2}{3}$.

3. Find the moment of inertia of a homogeneous right cone of altitude h and diameter $2a$ with respect to an axis through the vertex parallel to the base. *Ans.* $I = \frac{3m}{5}\left(\frac{a^2}{4} + h^2\right)$.

4. Solve Problem 12 of Art. 136 by means of plate elements.

5. Find the square of the radius of gyration of a square right pyramid with respect to an axis through the vertex parallel to one edge of the base.

Ans. $k^2 = \frac{3}{5}\left(\frac{b^2}{12} + h^2\right)$.

For any right pyramid or cone, the square of the radius of gyration with respect to an axis through the vertex parallel to the base is given by the equation,

$$k^2 = \frac{3}{5}(k_b^2 + h^2), \tag{4}$$

in which k_b is the radius of gyration of the base (considered as a thin plate) with respect to an axis through its center of gravity parallel to the axis through the vertex.

6. For any right cylinder or prism, show that the radius of gyration with respect to an axis perpendicular to the length through the center of one end is given by the equation,

$$k^2 = \frac{l^2}{3} + k_b^2.$$

7. For any axis perpendicular to the length of any right cylinder or prism, show that the radius of gyration is given by the equation,

$$k^2 = k_1^2 + k_b^2,$$

in which k_1 is the radius of gyration of a thin rod of length equal to that of the cylinder or prism with respect to a perpendicular axis, and k_b is the radius of gyration of a thin plate equivalent to a section of the cylinder or prism parallel to the base.

8. By means of Problem 7, find the moment of inertia of a homogeneous cylinder 10 inches long, 4 inches in diameter, and weighing 24 pounds, with respect to an axis perpendicular to its length and tangent to one end.

Ans. $I = 920$.

9. A solid cylinder, 10 inches in diameter and 12 inches long, has a cylindrical hole, 6 inches in diameter and 8 inches long, drilled in the upper end. The axis of the hole is parallel to the axis of the cylinder and 1 inch from it. Find the square of the radius of gyration of the remainder with respect to an axis in the upper end through the axis of the hole and perpendicular to the plane of the two axes.

141. Moment of Inertia of Connected Bodies.—To find the moment of inertia of two or more connected bodies with respect to a single axis, it is necessary to find the moment of inertia of each body with respect to a parallel axis through its center of gravity and then transfer to the common axis. The moment of inertia of all the bodies with respect to this axis is found by adding the moment of inertia of each of the separate parts. Since moment of inertia involves the square of a distance, it is always positive.

Problems

1. A solid sphere is 4 inches in diameter and weighs 10 pounds. It is hung on the end of a rod 1 inch in diameter and 40 inches long, which weighs 9 pounds. The end of the rod is tangent to the surface of the sphere. Find the radius of gyration with respect to a diameter of the rod at the end opposite the sphere. *Ans.* $k = 34.37$ in.

2. What change will be made in the result of Problem 1 if the diameter of the rod is neglected?

3. A right cone, 4 inches in diameter and 12 inches high, stands on the end of a cylinder, 2 inches in diameter and 20 inches long. The axis of the cone coincides with the axis of the cylinder. The cone and cylinder have the same density. Find the radius of gyration with respect to a diameter of the cylinder at the end opposite the cone. *Ans.* 17.77 in.

142. Summary.—Moment of inertia is defined *mathematically* by the equations,

$$I = \int r^2 dm; \; I = \Sigma mr^2; \qquad \text{Formula XI}$$

in which I is the moment of inertia, m is a finite element of mass, dm is an infinitesimal element of mass, and r is the distance of the element from the axis with respect to which the moment of inertia is taken.

The radius of gyration is defined *mathematically* by the equation

$$k^2 = \frac{I}{m}, \qquad \text{Formula XIII}$$

in which k is the radius of gyration and m is the entire mass of the body. The radius of gyration is the distance from the axis at which all the material of the body must be placed in order to have a moment of inertia equal to that of the actual body.

When the moment of inertia of a sphere of radius a is taken with respect to a diameter,

$$k^2 = \frac{2a^2}{5}.$$

When the moment of inertia of a solid cylinder of radius a is taken with respect to the axis of the cylinder,

$$k^2 = \frac{a^2}{2}.$$

When the moment of inertia of a right cone is taken with respect to the axis of the cone,

$$k^2 = \frac{3a^2}{10}.$$

When the moment of inertia of a rectangular parallelopiped is taken with respect to an axis through the center perpendicular to the edges of length b and d,

$$k^2 = \frac{b^2 + d^2}{12}.$$

When the moment of inertia of a right cylinder is taken for an axis through the center perpendicular to its length,

$$k^2 = \frac{l^2}{12} + \frac{a^2}{4}.$$

For any right cylinder or prism, for which the moment of inertia is taken with respect to an axis perpendicular to its length, the square of the radius of gyration is equal to the square of the radius of gyration regarded as a thin rod plus the square of the radius of gyration regarded as a thin plate.

To transfer from an axis through the center of gravity to a parallel axis,

$$k^2 = k_0^2 + d^2, \qquad \text{Formula XV}$$

in which k_0 is the radius of gyration with respect to the axis through the center of gravity and d is the distance between the axes.

143. Miscellaneous Problems

1. A homogeneous hollow cylinder has an inside diameter of 6 inches and and outside diameter of 10 inches. Find its radius of gyration with respect to an axis 12 inches from its axis and parallel to it.

Ans. $k = 13.34$ in.

2. A solid of revolution is formed by revolving the area enclosed by the X axis, the curve $y^2 = 2x$, and the line $x = 8$ about the X axis. Find the radius of gyration of this solid with respect to the X axis.

Ans. $k = 2.309$ in.

3. Find the radius of gyration of the solid of Problem 2 with respect to the Y axis.

4. A homogeneous solid cylinder, 2 inches in diameter and 20 inches long, is attached to a homogeneous solid hemisphere 8 inches in diameter. The center of the hemisphere coincides with the center of one end of the cylinder. If the density of the hemisphere is the same as the density of the cylinder, find the radius of gyration with respect to a diameter of the cylinder at the end opposite the hemisphere.

5. Solve Problem 4 if the density of the hemisphere is twice that of the cylinder.

CHAPTER XV

MOMENT OF INERTIA OF A PLANE AREA

144. Definition.—In calculating the strength and stiffness of members of structures and machines, much use is made of the expression $\int r^2 dA$, in which dA is an element of the area of a plane cross-section of the member, and r is the distance of the element from an axis in the plane or from an axis normal to the plane.

When the axis is *normal* to the plane, the integral is equivalent to the moment of inertia of a prism of cross-section equal to that of the member and of such length and density that its mass per unit area is unity. When the axis is *in* the plane of the section, the integral is equivalent to the moment of inertia of a plate so thin that the square of its thickness is negligible and of such density that its mass per unit area is unity. On account of the mathematical similarity with

$$I = \int r^2 dm,$$

the expression $\int r^2 dA$ is called the moment of inertia of the area.

Considered physically, the so-called moment of inertia of an area is entirely different from the moment of inertia of a solid. Considered mathematically, they are so much alike that it is worth while to treat them together. A formula which applies to a solid of uniform thickness may also be applied to the moment of inertia of a plane area by the substitution of area for mass.

The radius of gyration of a plane area is defined by the equation

$$k^2 = \frac{\textit{moment of inertia}}{\textit{area}}. \tag{1}$$

The formula for the transfer of axis for the moment of inertia of a plane area is

$$I = I_0 + Ad^2. \tag{2}$$

145. Polar Moment of Inertia.—When the axis is normal to the plane of the area, $\int r^2 dA$ is called the *polar* moment of inertia of the area. The polar moment of inertia of a plane area is

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equivalent to the moment of inertia of a right prism or cylinder for an axis parallel to its length, provided the density and length are such that the mass per unit area is unity.

It is customary to represent the polar moment of inertia by J.

$$J = \int r^2 dA. \qquad \text{Formula XV}$$

In Fig. 217, O represents an axis perpendicular to the plane of the paper. The element of area is $rd\theta dr$ and

$$J = \iint r^2 d\theta dr. \qquad (1)$$

In Fig. 218, the element of area is $dxdy$ and

$$J = \int r^2 dA = \iint (x^2 + y^2) dxdy. \qquad (2)$$

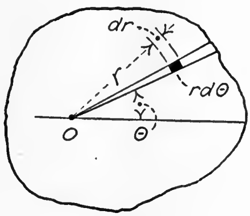

FIG. 217.

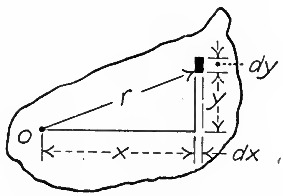

FIG. 218.

Problems

1. Find the polar moment of inertia of a circle of radius a with respect to an axis through its center and compare the moment of inertia and radius of gyration with those of a cylinder.

Ans. $J = \frac{\pi a^4}{2} = \frac{Aa^2}{2}$; $k^2 = \frac{a^2}{2}$.

2. Find the polar moment of inertia of a rectangle of breadth b and depth d with respect to an axis through its center.

Ans. $J = \frac{bd^3 + b^3d}{12} = \frac{A(b^2 + d^2)}{12}$; $k^2 = \frac{b^2 + d^2}{12}$.

3. Find the polar moment of inertia of an isosceles triangle of base b and altitude h with respect to an axis through the vertex.

Ans. $J = \frac{bh^3}{4} + \frac{b^3h}{48}$.

146. Axis in Plane.—When $\int r^2 dA$ is taken with respect to an axis in the plane of the area, it is called simply the *moment of inertia* of the area. This moment of inertia is a factor in the strength and stiffness of beams and columns, and has much greater application than the polar moment of inertia. The values of this moment of inertia for the common geometrical figures and for sections of the ordinary structural shapes are given in the handbooks of the steel companies.

If the moment of inertia is taken with respect to the X axis, $r = y$, and

$$I = \int y^2 dA = I_x. \tag{1}$$

If the moment of inertia is taken with respect to the Y axis, $r = x$, and

$$I = \int x^2 dA = I_y. \tag{2}$$

For convenience, the moment of inertia with respect to the X axis may be designated by I_x and the moment of inertia with respect to the Y axis by I_y. (This convention is convenient. It is not, however, in general use.)

Example

Find the moment of inertia of a circular area of radius a with respect to a diameter.

Figure 219 applies to this example. The moment of inertia is taken with respect to the X axis and the element of area is given in polar coördinates.

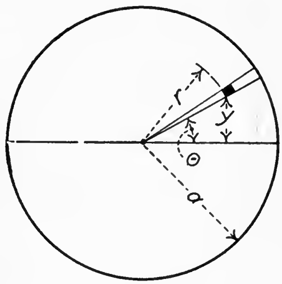

FIG. 219.

$$I = \int y^2 dA = \iint r^3 \sin^2 \theta d\theta dr = \left[\frac{r^4}{4}\right]_0^a \int \sin^2 \theta d\theta;$$

$$I = \frac{a^4}{4}\int \sin^2 \theta d\theta = \frac{a^4}{4}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi} = \frac{\pi a^4}{4} = \frac{Aa^2}{4}.$$

Problems

1. Find the radius of gyration of a circular area with respect to a diameter.

Ans. $k = \frac{a}{2}$.

2. Find the moment of inertia of a rectangle of base b and altitude d with respect to a line through the center parallel to the base.

Ans. $I = \frac{bd^3}{12}$.

3. Find the moment of inertia of the rectangle of Problem 2 with respect to the base. Solve by integration and also by transfer of axis.

Ans. $I = \frac{bd^3}{3}$.

4. The base of a rectangle is 6 inches and its altitude is 10 inches. Find its moment of inertia with respect to an axis 4 inches below the base and parallel to it. Solve two ways.

Ans. $I = \frac{6 \times 1000}{12} + 60 \times 81 = 5360 \text{ in.}^4$;

$I = \frac{6 \times 2744}{3} - \frac{6 \times 64}{3} = 5360 \text{ in.}^4$

5. By integration find the moment of inertia of a triangle of base b and altitude h with respect to an axis through the vertex parallel to the base.

Ans. $I = \frac{bh^3}{4}$.

6. Using the answer of Problem 5, find the moment of inertia of a triangle with respect to an axis through the center of gravity parallel to the base, and find the moment of inertia with respect to the base.

Ans. $I_0 = \frac{bh^3}{36}$; $I = \frac{bh^3}{12}$.

7. Find the moment of inertia of a triangle with respect to the base by subtracting the moment of inertia of an inverted triangle from the moment of inertia of a parallelogram.

8. A trapezoid has a lower base of 12 inches, an upper base of 6 inches, and an altitude of 9 inches. Find its moment of inertia with respect to the lower base. Solve by means of two triangles and check by means of a triangle and a rectangle.

Ans. $I = 1822.5$ in.^{4}

9. Find the moment of inertia of a 6-inch by 5-inch by 1-inch angle section with respect to an axis through the center of gravity parallel to the 5-inch leg. (Fig. 220.)

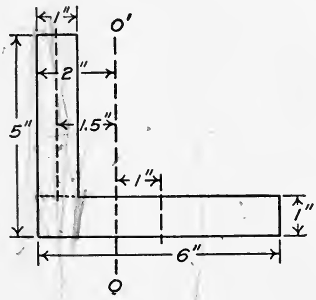

FIG. 220.

The center of gravity is found to be 2 inches from the back of the 5-inch leg. To find the moment of inertia with respect to the axis OO', the section is divided into a 6-inch by 1-inch rectangle and a 1-inch by 4-inch rectangle. The moment of inertia of each rectangle is found with respect to a vertical line through its center of gravity, and then transferred to the axis OO'.

The moment of inertia of the 6-inch by 1-inch rectangle with respect to the vertical line through its center of gravity is 18 in.^{4} This axis is 1 inch from the axis OO'. The moment of inertia of the 6-inch by 1-inch rectangle with respect to OO' is $18 + 6 \times 1^2 = 24$ in.^{4} The moment of inertia of the 1-inch by 4-inch rectangle with respect to OO' is $0.33 + 4 \times 1.5^2 = 9.33$. The moment of inertia of the entire section with respect to OO' is $24 + 9.33 = 33.33$ in.^{4}

The problem may also be solved by finding first the moment of inertia of the entire section with respect to some vertical axis and then by transferring to the parallel axis through the center of gravity. The moment of inertia with respect to the back of the 5-inch leg is

$$I = \frac{1 \times 6^3}{3} + \frac{4 \times 1^3}{3} = 73\tfrac{1}{3}.$$

This axis is 2 inches from the axis OO'.

$$I_0 = 73\tfrac{1}{3} - 10 \times 2^2 = 33\tfrac{1}{3}.$$

10. Solve Problem 9 by dividing the section into two equal rectangles as shown in Fig. 221. For the second method, find I with respect to the axis CC' and then transfer to OO' through the center of gravity.

11. Find the moment of inertia of the 6-inch by 5-inch by 1-inch angle section of Fig. 220 with respect to the axis through the center of gravity parallel to the 6-inch leg. Calculate the radius of gyration and compare the results with those of Cambria Steel or Carnegie Pocket Companion for a similar smaller section.

12. Figure 222 shows a standard channel section without fillets or curves. The slope of the flange of standard section is $\frac{1}{6}$. Find the moment of inertia of this section with respect to an axis through the center of gravity perpendicular to the web.

$$Ans. \quad I = \frac{bd^3}{12} - \frac{h^4 - l^4}{16}.$$

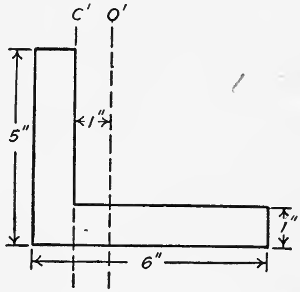

FIG. 221.

13. Take the dimensions from a steel handbook and calculate the moment of inertia of a 12-inch 20.5-pound channel section. Compute the area of the section and the radius of gyration.

14. With the moment of inertia of a 12-inch 20.5-pound channel known, calculate the moment of inertia of a 12-inch 30-pound channel and a 12-inch 40-pound channel.

15. Find the distance of the center of gravity of a 12-inch 20.5-pound channel from the back of the web. Find the moment of inertia with respect to the back of the web and transfer to the parallel axis through the center of gravity. Compare results with the handbook.

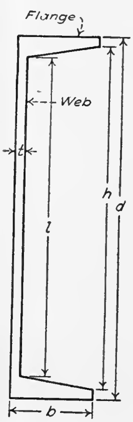

FIG. 222.

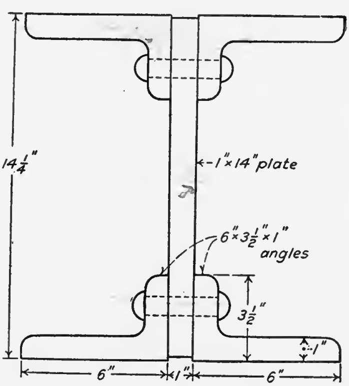

FIG. 223.

16. Look up the expression for the moment of inertia of an I-beam section for axis perpendicular to the web. Derive the expression. Do the same for the axis through the center parallel to the web.

17. Find the moment of inertia and the radius of gyration of a 15-inch 42-pound I-beam for axes through the center of gravity perpendicular to the web and parallel to the web. Compare the results with the handbook.

18. Figure 223 gives the dimensions of a plate-and-angle section made up of one 14-inch by 1-inch plate and four 6-inch by $3\frac{1}{2}$-inch by 1-inch angles. Look up the moment of inertia and the location of the center of gravity of one of these angles in the handbook and compute the moment of inertia of the section with respect to an axis through the center of the section perpendicular to the plate and for an axis parallel to the plate. Calculate the radius of gyration and compare with the handbook.

19. Two 10-inch 15-pound channels are placed vertical and riveted to a 12-inch by ½-inch plate at the top. Find the moment of inertia with respect to an axis through the center of gravity perpendicular to the channels. Use all the data you can get from the handbook. Where is such a section used?

20. A circle of radius a has a segment cut off by means of a chord at a distance $\frac{a}{2}$ from the center. By integration find the moment of inertia of the area between the chord and the circle with respect to the chord as an axis.

21. Find the moment of inertia of the area between the curves $y^2 = 6x$ and $x^2 = 6y$ with respect to the X axis.

22. Find the moment of inertia of the area bounded by the curve $xy = 24$, the line $x = 2$, and the line $y = 3$ with respect to the X axis.

23. Find the moment of inertia of the area bounded by the curve $y = x^2$, the line $x = 4$ and the X axis with respect to the line $x = -2$.

147. Relation of Moments of Inertia.—If dA is an element of area in the XY plane, the moment of inertia of the entire area with respect to the X axis is

$$I_x = \int y^2 dA. \tag{1}$$

With respect to the Y axis

$$I_y = \int x^2 dA. \tag{2}$$

The polar moment of inertia with respect to an axis through the origin is

$$J = \int r^2 dA = \int (x^2 + y^2) dA = \int x^2 dA + \int y^2 dA = I_y + I_x. \tag{3}$$

The sum of the moments of inertia of a plane area with respect to two axes in its plane which are perpendicular to each other is equal to the polar moment of inertia of the area with respect to the intersection of these axes. It follows that the sum of the moments of inertia of a plane area with respect to any pair of axes in its plane which are at right angles to each other and pass through the same point is constant.

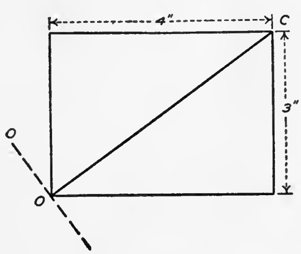

FIG. 224.

Example

Find the moment of inertia of a 4-inch by 3-inch rectangle with respect to a line through one corner perpendicular to the diagonal.

The diagonal OC of Fig. 224 divides the rectangle into two triangles, each of which has a base of 5 inches and an altitude of 2.4 inches. The moment of inertia of each triangle with respect to OC is 5.76 in.^{4} and the moment of inertia of the rectangle is 11.52 in.^{4} If moments of inertia are taken with respect to horizontal and vertical axes through O,

$$I_x = 36;\ I_y = 64;$$
$$I_x + I_y = J = 11.52 + I;$$
$$I = 88.48 \text{ in.}^4$$

Problems

1. The moment of inertia of a circle with respect to a diameter is $\frac{\pi a^4}{4}$. By the principle of Equation (3) show that the polar moment of inertia with respect to the center is $\frac{\pi a^4}{2}$.

2. The polar moment of inertia of a circle with respect to a point at its circumference is $\frac{3\pi a^4}{2}$. Find the moment of inertia of a circle with respect to a tangent in its plane. Check by the transfer of axis.

3. Find the moment of inertia of a 6-inch by 6-inch by 1-inch angle section with respect to the axis which bisects the angle between the legs of the section. Use Equation (3) and get all the data from the handbook. Check by completing the square and subtracting the moment of inertia of the additional area from the moment of inertia of the square.

148. Product of Inertia.—To find the moment of inertia of a plane figure with respect to some inclined axis in its plane often involves difficult integrations or limits. These difficulties may be avoided by finding the moment of inertia with respect to a pair of axes for which the calculations are easiest and then transferring to a new axis at the required angle with these axes. In making this transformation (Art. 150) one of the terms is found to be $\int xydA$. This integral is called the *product of inertia of the area*. The product of inertia is represented by H in algebraic equations.

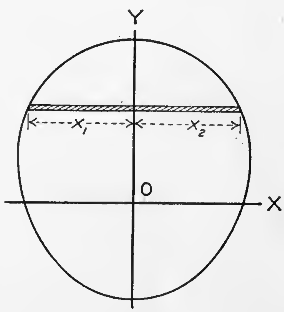

FIG. 225.

$$H = \int xydA \qquad (1)$$

The product of inertia has no physical significance. It is, however, so useful in the calculation of the moment of inertia of plane areas that it is desirable to master some of its properties.

If an area has an axis of symmetry, the product of inertia with respect to that axis and the axis perpendicular to it is zero. In Fig. 225, the Y axis is an axis of symmetry. Any horizontal line extends equal distances on each side of the Y axis. If Equation (1) is integrated first with respect to x

$$H = \int xydA = \int\int xydydx = \frac{1}{2}\int\Big[x^2\Big]_{x_1}^{x_2} ydy = \frac{1}{2}\int\Big[x_2^2 - x_1^2\Big]\, ydy.$$

When the area is symmetrical with respect to the Y axis, x_1 is numerically equal to x_2 but has the opposite sign. Since the square of a negative quantity is positive, x_1^2 has the same sign as x_2^2 and $x_2^2 - x_1^2 = 0$. Consequently

$$H = 0. \tag{2}$$

Problems

1. By integration find the product of inertia of a rectangle of width b and height d with respect to the lower edge and the left edge as axes.

Ans. $H = \frac{b^2d^2}{4}$.

2. Solve Problem 1 with respect to the lower edge and the right edge as axes. *Ans.* $H = -\frac{b^2d^2}{4}$.

3. Find the product of inertia of the first quadrant of a circle of radius a with respect to the X and Y axes. *Ans.* $H = \frac{a^4}{8}$.

4. Find the product of inertia with respect to the coördinate axes of the area bounded by these axes and the line whose intercepts are $x = 6$ and $y = 4$. *Ans.* $H = 24$ in.^{4}

5. Find the product of inertia with respect to the coördinate axes of the area bounded by these axes and the line whose intercepts are $(0, d)$ and $(b, 0)$.

149. Transfer of Axes for Product of Inertia.—In Fig. 226, OX and OY are a pair of axes at right angles to each other, and $O'X'$ and $O'Y'$ are a second pair of axes parallel to these. The coördinates of the point O with respect to the axes $O'X'$ and $O'Y'$ are a and b. It is required to find the product of inertia of the area with respect to $O'X'$ and $O'Y'$.

The coördinates of an element dA with respect to OX and OY are x and y. With respect to $O'X'$ and $O'Y'$, the coördinates are $a + x$ and $b + y$. The product of inertia is given by the equation,

$$H = \int(a + x)(b + y)dA = ab\int dA + a\int ydA + b\int xdA + \int xydA. \quad (1)$$

The first term, $ab\int dA$, is equivalent to abA. The last term $\int xydA$ is H_0, which is the product of inertia with respect to the axes OX and OY. The term $\int ydA$ is the moment of the area with respect to the X axis and the term $\int xdA$ is the moment of the area with respect to the Y axis. If the point O is at the center of gravity of the area, these terms vanish and

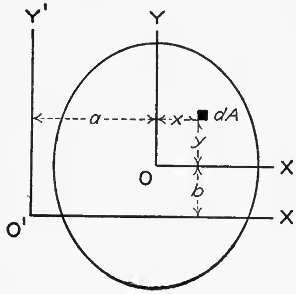

FIG. 226.

$$H = H_0 + abA \qquad \text{Formula XVI}$$

If either OX or OY is an axis of symmetry, the last term also vanishes. While the moment of inertia is always positive, the product of inertia may be either positive or negative.

Problems

1. Find the product of inertia of a rectangle 6 inches wide and 4 inches high with respect to the lower edge and the left edge as axes.

Ans. $H = 3 \times 2 \times 24 = 144$ in.^{4}

2. Find the product of inertia of the rectangle of Problem 1 with respect to the lower edge and the right edge as axes.

Ans. $H = (-3)2 \times 24 = -144$ in.^{4}

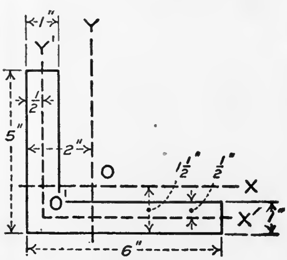

FIG. 227.

3. Find the product of inertia of a semicircle in the first and fourth quadrants with respect to the diameter which bounds it and the tangent at the lower end of this diameter.

Ans. $H = \dfrac{2a^4}{3}$.

4. Find the product of inertia of a 6-inch by 5-inch by 1-inch angle section in the position of Fig. 227 with respect to the back of the legs as axes.

5. Find the product of inertia of the angle section of Problem 4 with respect to axes through the center of gravity parallel to the legs.

The section may be divided into two rectangles. The product of inertia may be transferred from lines through the center of gravity of each of these rectangles to the parallel lines through the center of gravity of the figure.

Another way is to take the product of inertia with respect to the axes $O'X'$ and $O'Y'$ which bisect the legs. The product of inertia for these axes is zero, since $O'X'$ is an axis of symmetry for the horizontal leg and $O'Y'$ is an axis of symmetry for the vertical leg. Transferring to the center of gravity,

$$0 = H_0 + abA = H_0 + 1.5 \times 1 \times 10$$
$$H_0 = -15 \text{ in.}^4$$

150. Change of Direction of Axis for Moment of Inertia.—In Fig. 228, OX and OY are a pair of axes for which the moment of inertia is known. These are $I_x = \int y^2dA$ and $I_y = \int x^2dA$. It is desired to find the moment of inertia with respect to the axis OX' which makes an angle θ with OX.

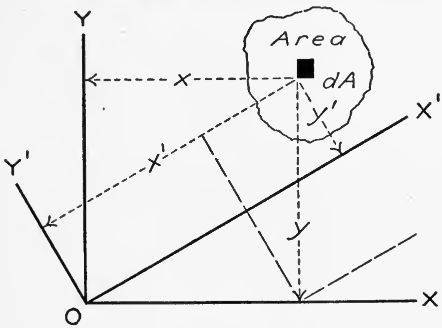

FIG. 228.

Let OY' be an axis perpendicular to OX'. The coördinates of the element dA with respect to OX' and OY' as axes are (x', y'). The moment of inertia with respect to OX' is

$$I' = \int y'^2dA. \tag{1}$$

From the geometry of the figure,

$$y' = y \cos \theta - x \sin \theta; \tag{2}$$
$$I' = \int (y^2 \cos^2 \theta - 2xy \cos \theta \sin \theta + x^2 \sin^2 \theta)dA; \tag{3}$$
$$I' = I_x \cos^2 \theta + I_y \sin^2 \theta - 2 \cos \theta \sin \theta \int xydA; \tag{4}$$
$$I' = I_x \cos^2 \theta + I_y \sin^2 \theta - H \sin 2\theta; \tag{5}$$
$$I' = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - H \sin 2\theta. \tag{6}$$

Problems

1. Find the moment of inertia of a 6-inch by 4-inch rectangle with respect to an axis through the center at an angle of 25 degrees with the 6-inch edges.

$I_x = 32$, $I_y = 72$, $H = 0$;

$I' = 52 - 20 \cos 50° = 52 - 20 \times 0.6428 = 39.144$ in.^{4}

2. Find the moment of inertia of the rectangle of Problem 1 with respect to the diagonal by means of Equation (6). Check by dividing the rectangle into two triangles and finding the moment of inertia of each with respect to the diagonal as the base.

3. Find the moment of inertia of the rectangle of Problem 1 with respect to an axis through one corner. The axis makes an angle of 20 degrees with the 6-inch edge, and cuts one 4-inch edge.

Ans. $I' = 208 - 61.28 - 92.56 = 54.16$ in.^{4}

151. Change of Direction of Axes for Product of Inertia.—From Fig. 228, the product of inertia with respect to the axes OX' and OY' is

$$H' = \int x'y'dA; \tag{1}$$
$$H' = \int (x \cos \theta + y \sin \theta)(y \cos \theta - x \sin \theta)\, dA; \tag{2}$$
$$H' = (\cos^2 \theta - \sin^2 \theta)\int xydA + \cos \theta \sin \theta \int (y^2 - x^2)dA; \tag{3}$$
$$H' = H \cos 2\theta + \frac{I_x - I_y}{2} \sin 2\theta. \tag{4}$$

If H' is zero the second member of Equation (4) is zero, and

$$H \cos \theta = \frac{I_y - I_x}{2} \sin 2\theta; \tag{5}$$
$$\tan 2\theta = \frac{2H}{I_y - I_x}. \tag{6}$$

Since the tangent of an angle may have any value from minus infinity to plus infinity, there is always some direction for which the product of inertia for a pair of axes through any point is zero. If there is a line of symmetry through the point, this line is one of the axes for which the product of inertia is zero.

Problems

1. Find the direction of the axes through the lower left corner of a rectangle 6 inches wide and 4 inches high for which the product of inertia is zero. (Fig. 229.)

Ans. 30° 28′ and 120° 28′ with the 6-inch edge.

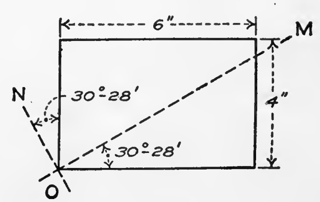

FIG. 229.

2. Find the direction of the axes through the center of gravity of a 6-inch by 4-inch by 1-inch angle section for which the product of inertia is zero.

Ans. Tan $2\theta = -1$; $\theta = -22° 30'$ and $67° 30'$.

152. Maximum Moment of Inertia.—To find the direction of the axis through any given point for which the moment of inertia is greater than for any other axis through that point, the mathematical conditions for maximum and minimum are applied to Equation (6) of Art. 150.

$$I' = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - H \sin 2\theta. \quad (1)$$

Differentiating with respect to θ,

$$\frac{dI}{d\theta} = (I_y - I_x) \sin 2\theta - 2H \cos 2\theta. \quad (2)$$

Equating the derivative to zero and solving for θ,

$$\tan 2\theta = \frac{2H}{I_y - I_x}. \quad (3)$$

Equation (3) is identical with Equation (6) of Art. 151. It is evident, therefore, that the directions for which the moment of inertia is a maximum or a minimum coincide with the two directions for which the product of inertia is zero.

Two angles which are 180 degrees apart have the same tangent. Equation (3), therefore, gives two values of θ, which differ by 90 degrees. For one of these directions, the moment of inertia is a maximum; for the other, the moment of inertia is a minimum.

If an axis of symmetry passes through a given point, the moment of inertia for that axis is either a maximum or a minimum.

Problems

1. Find the maximum and minimum moment of inertia of a 6- inch by 4- inch rectangle for axes through the lower left corner. (Fig. 229.)

From Problem 1 of Art. 151, $\tan 2\theta = 1.800$; $2\theta = 60° \, 57'$ and $240° \, 57'$.
$I_{min} = 208 - 80 \cos 60° \, 57' - 144 \sin 60° \, 57' = 43.27$ in.^{4}
$I_{max} = 208 - 80 \cos 240° \, 57' - 144 \sin 240° \, 57' = 372.73$ in.^{4}
I_{min} is the moment of inertia with respect to OM of Fig. 229.

2. Find the least moment of inertia and the least radius of gyration of a 6-inch by 6-inch by 1-inch angle section. Take I_x and I_y from the handbook.

3. Find the maximum and minimum moments of inertia for axes through the center of gravity of a 6-inch by 4-inch by 1-inch angle section. Take I_x and I_y from handbook. Calculate the least radius of gyration and compare the result with the handbook.

153. Center of Pressure.—In Art. 104, the center of pressure of a liquid on a plane surface was defined as the point of application of the resultant of all the pressure on one side of the surface. Only the simplest area could be conveniently calculated by the methods given there.

In Fig. 230, dA is an element of a vertical surface subjected to liquid pressure on one side. All parts of the element are at a distance y below the surface of the liquid. If w is the weight of the liquid per unit volume,

$$\text{horizontal pressure on } dA = wydA. \qquad (1)$$

The moment of the pressure about the line of intersection of the plane of the surface with the surface of the liquid is

$$dM = wy^2dA; \qquad (2)$$
$$M = w\int y^2dA. \qquad (3)$$

The integral of Equation (3) is the moment of inertia of the area with respect to the line of intersection with the surface of the liquid.

$$M = wI. \qquad (4)$$

If y_c is the distance of the center of pressure from the surface of the liquid, and if $\bar{y}$ is the distance of the center of gravity of the area from the surface of the liquid, then

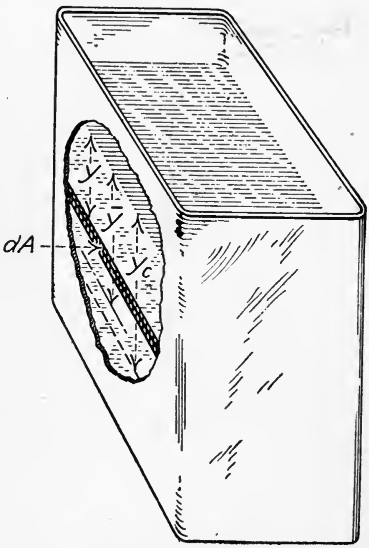

FIG. 230.

$$y_c = \frac{\text{moment}}{\text{total pressure}} = \frac{wI}{w\bar{y}A} = \frac{k^2}{\bar{y}}, \qquad (5)$$

in which k is the radius of gyration of the area with respect to the line of intersection of its plane with the surface of the liquid

Problems

1. Find the center of pressure of a vertical rectangular gate of width b and height h when the liquid reaches the top.

$$\textit{Ans.} \quad y_c = \frac{I}{\bar{y}A} = \frac{bh^3}{3} \div \frac{bh^2}{2} = \frac{2h}{3}.$$

2. A vertical rectangular gate is 6 feet wide and 4 feet high and is subjected to the pressure of water which rises 6 feet above the top of the gate. Find the center of pressure. *Ans.* 8.167 ft. below the surface of the water.

If k_0 is the radius of gyration of the surface with respect to the horizontal axis in its plane through the center of gravity,

$$k^2 = k_0^2 + \bar{y}^2, \tag{6}$$

and Equation (5) becomes

$$y_c = \frac{\bar{y}^2 + k_0^2}{\bar{y}} = \bar{y} + \frac{k_0^2}{\bar{y}}. \tag{7}$$

The distance of the center of pressure below the center of gravity of the area is equal to the square of the radius of gyration of the area divided by the distance of its center of gravity below the surface of the liquid.

Problems

3. A vertical circular gate of radius r is subjected to the pressure of liquid which just reaches the top. Find the center of pressure.

Ans. $\frac{r}{4}$ below the center of the circle.

4. A vertical circular gate, 4 feet in diameter, is subjected to the pressure of a liquid which reaches 10 feet above the center. Locate the center of pressure *Ans.* 0.1 ft. below the center.

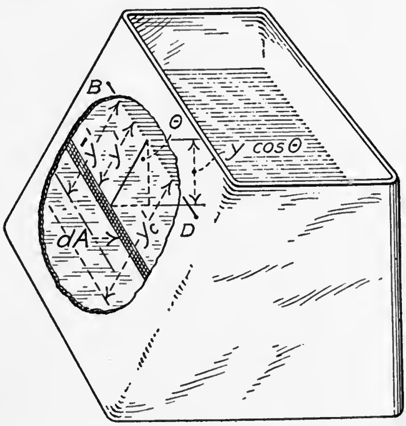

FIG. 231.

5. A vertical circular gate, 6 feet in diameter, is subjected to the pressure of water which rises to the center. The gate is held by two bolts at opposite ends of the horizontal diameter and a single bolt at the bottom. Find the tension in each bolt.

Ans. 662.7 lb. in the lower bolt.

6. A solid cylinder, 1 foot in diameter, is cut by two planes. One plane is perpendicular to the axis and the other plane cuts the cylinder so that the longest element is 2 feet long and the shortest element is 1 foot long. Use the equations above to find the center of gravity of the volume between the planes.

Ans. 0.5 inch from the axis.

Figure 231 shows liquid pressure on an inclined surface. The line BD is the intersection of the plane of the inclined surface with the surface of the liquid. The element dA is at a distance y' from the line BD. If θ is the angle which the plane makes with the vertical, the vertical distance of the element of area below the surface of the liquid is $y' \cos \theta$, and the pressure on the element is $wy' \cos \theta dA$. The moment equation about BD is

$$M = w \cos \theta \int y'^2 dA = WI' \cos \theta. \qquad (8)$$

in which I' is the moment of inertia of the area with respect to the axis BD.

$$y_c' = \frac{wI' \cos \theta}{w\bar{y}'A \cos \theta} = \frac{I'}{\bar{y}'A}. \qquad (9)$$

Equation (9) is the same as Equation (5) except that the distances are not vertical. An equation may be written which is similar to Equation (7).

154. Moment of Inertia by Moment of a Mass.—In the preceding articles it was shown that the moment of a pressure which varies as the distance from the axis of moments may be calculated by means of the moment of inertia of the plane area. From Equation (5) of Art. 153

$$wI = w\bar{y}Ay_c; \qquad (1)$$

$$I = \bar{y}Ay_c. \qquad (2)$$

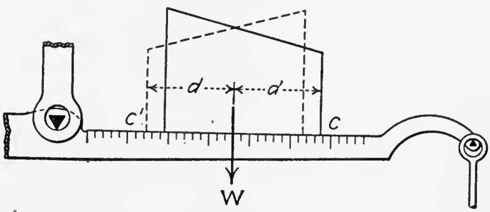

FIG. 232.

From Equation (2), the moment of inertia of a plane area with respect to a given axis is the product of the area A, multiplied by the distance of its center of gravity from the axis, and by the distance of the center of a pressure which varies as the distance from the axis. The center of pressure is located at the *center of gravity* of a solid, the base of which is the given area and the height is proportional to the distance from the axis.

The center of gravity of a solid may be found experimentally by moments. Figure 232 shows a convenient method. The body whose center of gravity is desired is placed on a balance beam and carefully balanced with the beam in a horizontal position. It is then turned 180 degrees about a vertical axis to the position shown by the broken lines of the figure, and moved along the beam until exact balance is obtained with no change of the weights. The center of gravity is then at the same location as it was in the first position. If C is a point in the body, and C' is the same point after the body is turned, the center of gravity is located in a vertical line midway between C and C'.

Figure 233 represents a section of a beam whose moment of inertia is desired for a line through the center of gravity parallel to a given line OO'. If the section is large, the outlines may be traced on paper and the area determined by a planimeter. If the section is small, a portion of the beam of convenient length may be cut off between two sections. The volume of this portion may be determined by weighing in air and in water and its area may then be calculated by dividing the volume by the length.

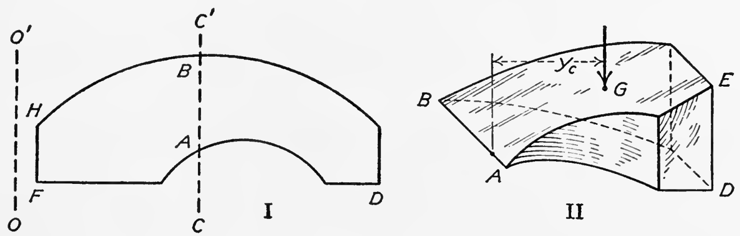

FIG. 233.

The center of gravity of the section may be found by balancing this portion of uniform thickness upon the beam of a balance, as in Fig. 232. The line CC', Fig. 233, passes through the center of gravity.

To get the moment of inertia, a force is required which varies directly as the distance from the center of gravity. Wedge-shaped pieces are cut from the beam by planes which intersect at AB. The plane ABD of Fig. 233, II, is normal to the length of the beam, and the plane ABE makes a convenient angle with the length. This wedge-shaped piece is balanced as in Fig. 232 and its center of gravity determined. The distance of the center of gravity from the line AB gives Y_c for this part of the section. The location of the center of gravity of the part to the right of AB is found by balancing a piece of uniform length. This experiment gives $\bar{y}$.

The product $y_cA\bar{y}$ for the part to the right of AB gives the moment of inertia of that part of the section. A similar set of experiments gives the moment of inertia of the part of the section to the left of AB. The moment of inertia of the entire section is the sum of these two results.

Problems

1. A portion of a beam, 0.25 inch in length, weighs 0.353 lb. in air and 0.309 lb. in water. Find the area of the section. *Ans.* $A = 4.88$ in.^{2}

2. The section of Problem 1 was 5 inches wide from a point F to a point D. It was placed on a balance beam with F to the left and balanced. When it was turned end for end and balanced again, it was found that the point F was 4.24 inches from the position of the previous balance. How far is the center of gravity from F? *Ans.* = 2.12 in.

3. The beam of Problems 1 and 2 was cut by two planes which intersected on the line FH of Fig. 233, I. One plane was normal to the length and the other made an angle with it. The portion was placed on the scale beam with the normal section horizontal and it was found that the center of gravity of the wedge was 3.21 inches from FH. Find the moment of inertia of the section with respect to the axis FH and with respect to the parallel axis through the center of gravity of the section.

Ans. $I = 33.22$ in.^{4}; $I_0 = 11.29$ in.^{4}

155. Summary.—The moment of inertia of a plane area is defined mathematically by the expression,

$$\text{Moment of inertia} = \int r^2 dA.$$

When the axis is perpendicular to the plane of the area, the moment of inertia is called the polar moment of inertia. Polar moment of inertia is represented by the letter J. The moment of inertia of a plane area with respect to an axis in its plane is represented by the letter I.

The sum of two moments of inertia with respect to axes in the plane of the area at right angles to each other is equal to the polar moment of inertia with respect to an axis through the intersection of these two axes.

$$J = I_x + I_y.$$

The formulas for the transfer to parallel axes are

$$I = I_0 + Ad^2; J = J_0 + Ad^2.$$

Radius of gyration is given by

$$k^2 = \frac{I}{A} \text{ or } k^2 = \frac{J}{A}.$$

Product of inertia is defined by the equation

$$H = \int xy dA.$$

Product of inertia is useful in changing the direction of the axis for moment of inertia, and in finding the direction of the axes for which the moment of inertia is a maximum or a minimum.

Moment of inertia is a maximum for one of the pair of axes for which the product of inertia is zero and a minimum for the other. Product of inertia is zero if either axis is an axis of symmetry. Moment of inertia is a maximum or minimum and product of inertia is zero for the directions which satisfy the equation

$$\tan 2\theta = \frac{2H}{I_y - I_x}$$

The location of the center of pressure of a liquid on a plane surface is given by the equations,

$$y_c = \frac{k^2}{\bar{y}} = \bar{y} + \frac{k_0^2}{\bar{y}},$$

in which y_c is the distance of the center of pressure from line of intersection of the plane of the area with the surface of the liquid, $\bar{y}$ is the distance of the center of gravity of the area from this line, k is the radius of gyration of the area with respect to the same line, and k_0 is the radius of gyration with respect to a parallel line through the center of gravity. These equations apply to inclined and to vertical surfaces.

CHAPTER XVI

MOTION

156. Displacement.—When a body changes its position relative to another body, it is said to be displaced relatively to the other body. The distance from the initial position to the final position is the *displacement* of the body. Displacement is generally measured relative to the earth. Displacement has direction as well as magnitude, and is, therefore, a vector quantity.

Problems

1. A man steps from a car and walks east a distance of 40 feet. During the same time, the car moves west 60 feet. What is the displacement of the man relative to the earth and relative to the car?

Ans. 40 feet east; 100 feet east.

2. A body moves from one position to another position which is 40 feet east and 30 feet north of the starting point. What is its displacement?

Ans. 50 feet north 53° 08′ east.

3. A base ball is struck by a bat. Two seconds later it is 80 feet above and 150 feet south of the starting point. What is the displacement?

157. Velocity.—Rate of displacement is called *velocity.* In problems of mechanics, velocity is generally expressed in feet per second. Since displacement is a vector, and velocity is displacement divided by time (which is not a vector) it follows that velocity is a vector. A velocity of ten feet per second north is not equivalent to a velocity of ten feet per second east. When only the magnitude of the motion is considered without reference to its direction, the rate of displacement is called *speed.* A body moving east at the rate of ten feet per second has the same speed as a body moving north at the rate of ten feet per second.

Example

A man is 40 feet south of a given point when his watch reads 1 min. 20 sec. When he is 100 feet south of the point, his watch reads 1 min. 35 sec. What is his average velocity during that interval?

The displacement is $100 - 40 = 60$ feet south. The time interval is 15 seconds. Average velocity south $= \frac{60}{15} = 4$ feet per second.

Velocity is calculated by dividing the difference in position by the corresponding difference in time. If x_1 is a coördinate of the position at the time t_1 and x_2 is a coördinate of the position at the time t_2, and if v_x is the component of the velocity parallell to the X axis,

$$v_x = \frac{x_2 - x_1}{t_2 - t_1}. \qquad \text{Formula XVII}$$

Similar expressions give the components of the velocity parallel to the other axes of coördinates. The actual velocity is the vector sum of these components.

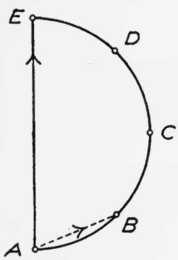

FIG. 234.

Formula XVII and the similar expressions give the *average velocity* during the interval of time. The actual velocity at any instant may be quite different. In Fig. 234, a body may start at A and move along the curved path $ABCDE$, passing over each interval in one second. If an interval of one second is considered, the velocity during the first second is equal to the length AB and is in the direction from A to B. If the entire 4 seconds are considered as the interval, the velocity is, apparently, one-fourth of the length AE and is in the direction from A to E. Again, the speed or magnitude of the velocity may change. A body may move 2 feet in one second, 5 feet in the next second, and 11 feet in the third second. For the first second, the average velocity is 2 feet per second; for the first two seconds, the average velocity is 3.5 feet per second; for the three seconds, the average velocity is 6 feet per second. When the velocity varies in direction and magnitude and it is desired to find the true velocity at any instant, a small displacement and a correspondingly small interval of time must be employed. Formula XVII becomes

$$v_x = \frac{\Delta x}{\Delta t}. \qquad (1)$$

To determine experimentally the actual velocity at any given instant, means must be devised for measuring small intervals of time and corresponding increments of displacement. Suppose it is desired to find the velocity of a steel ball attached to a small parachute after falling two seconds from rest. The ball may fall beside a vertical scale as in Fig. 235. The time and displacement may be measured by means of a photographic camera provided with an "instantaneous shutter" of known time. An electromagnetic arrangement may be provided which will release the shutter two seconds after the ball starts to fall. If the ball is properly illuminated, its displacement is given by a vertical streak, AB of Fig. 235. Suppose that the "time" of the shutter is 0.02 second, and that the length of the streak on the plate is equivalent to 0.36 foot of the scale. The average velocity for the 0.02 second is

$$v = \frac{0.36}{0.02} = 18 \text{ feet per second.}$$

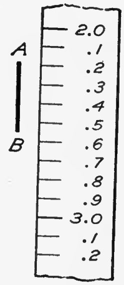

FIG. 235.

This experiment gives the average velocity for the interval of 0.02 second. If the change of velocity is all in one direction, that is, if the velocity is either increasing or decreasing throughout the entire interval, the average velocity is the velocity at about the middle of the interval. The result of this experiment is, then, the velocity at about 2.01 seconds after the beginning of the fall. If the velocity is desired at 2 seconds from the beginning of the fall, the apparatus must be arranged to release the shutter 1.99 seconds after the body starts to move.

Problems

1. The position of a body is given by the equation $x = 8t^2$, in which x is the position in feet and t is the time in seconds. Find the approximate velocity at the end of 4 seconds by means of the displacement during a 2-second interval. *Ans.* Average $v = \dfrac{8 \times 5^2 - 8 \times 3^2}{2} = 64$ ft. per sec.

2. Solve Problem 1 for the velocity at the end of 4 seconds by means of the average velocity during an interval of 0.2 second.

Ans. Average $v = \dfrac{8 \times 4.1^2 - 8 \times 3.9^2}{0.2} = 64$ ft. per sec.

3. The position of a body is given by $x = 2t^3$. Using an interval of 2 seconds, find the approximate velocity at the end of 4 seconds.

Ans. Average $v = \dfrac{2 \times 5^3 - 2 \times 3^3}{2} = 98$ ft. per sec.

4. Solve Problem 3 for an interval of 0.2 second.

Ans. Average $v = \dfrac{2(4.1^3 - 3.9^3)}{0.2} = 96.02$ ft. per sec.

In Problems 1 and 2, the average velocity for the 2-second interval is the same as the average velocity for the 0.2-second interval. It is fair to assume that the average velocity is the actual velocity at the middle of the interval and that 64 feet per second is the true velocity at the end of 4 seconds.

In Problems 3 and 4, the average velocity for the 2-second interval is greater than the average velocity for the 0.2-second interval. It is evident, therefore, that the average velocity is not exactly the velocity at the middle of the interval and that a very short interval must be used to get an exact result.

Problem

5. In Problem 3, let the interval be 0.02 second and find the approximate velocity at the end of 4 seconds.

$$\textit{Ans.}\quad v = \frac{2(4.01^3 - 3.99^3)}{0.02} = 96.0002 \text{ ft. per sec.}$$

It is evident that an interval of 0.2 second is sufficiently short to give a result as accurate as can be measured.

158. Average and Actual Velocity.—Formula XVII gives the average velocity for the interval. From the problems of the preceding article, it is evident that the average velocity is sometimes equal to the velocity at the middle of the time interval, and at other times is not equal to that velocity. A few cases will now be considered in order to determine under what conditions it is necessary to make the time interval very short.

Let $x = kt^2$ be the expression for the displacement, and let the average velocity be found for an interval of $2\Delta t$ seconds.

$$\text{Average } v = \frac{k(t + \Delta t)^2 - k(t - \Delta t)^2}{2\Delta t} = \frac{4kt\Delta t}{2\Delta t} = 2kt.$$

The result is independent of the length of the interval $2\Delta t$. Consequently, when the displacement varies as the square of the time, the velocity at the middle of the interval of time is equal to the average velocity during that interval. This is true also when the displacement varies as the time.

When the displacement varies as the cube of the time, $x = kt^3$. For an interval $2\Delta t$,

$$\text{average } v = \frac{k(t + \Delta t)^3 - k(t - \Delta t)^3}{2\Delta t} = \frac{6t^2\Delta t + 2\Delta t^3}{2\Delta t} = 3t^2 + \Delta t^2.$$

It is evident that the average velocity is slightly different from the velocity at the middle of the interval. The error, however, decreases rapidly with decrease of the interval of time.

In an experimental problem, it may be known that the displacement varies as the square of the time, as the first power of the time, or as a combination of the two. In these cases, the time interval may be taken of any convenient length. The average velocity during the interval is the velocity at the middle of the interval. If the experimental values of the velocity are the same, whether the intervals are long or short, it may be taken for granted that the displacement varies as the time or as the square of the time.

For any form of mathematical expression for the displacement in terms of the time, it is easy to calculate the effect of the length of the interval upon the accuracy of the result.

159. Velocity as a Derivative.—When the interval of time becomes infinitesimal, Formula XVII reduces to

$$v_x = \frac{dx}{dt}.$$

In Problem 1 of Art. 157,

$$x = 8t^2,$$

$$\frac{dx}{dt} = 16t.$$

When $t = 4$ seconds, $v_x = 16 \times 4 = 64$ ft. per sec.

In Problem 3 of Art. 157,

$$x = 2t^3,$$

$$\frac{dx}{dt} = 6t^2.$$

When $t = 4$ seconds, $v_x = 6 \times 16 = 96$ ft. per sec.

Since these derivatives give the correct result so easily, it may be asked why the methods of Article 157 are employed. The derivative *is* sufficient when the equation of the displacement in terms of the time is accurately known. In most cases, however, the displacement is not known in terms of the time and an experimental determination is required to get the velocity. The work of the two preceding articles is intended to show with what accuracy the average velocity during an interval gives the true velocity at the middle of that interval.

Problems

1. The position of a body is given by the equations $x = 3t^2$, $y = t^3$, in which the distances are expressed in feet and the time is given in seconds. Find the magnitude and direction of the resultant velocity when $t = 4$ seconds. *Ans.* $v = 53.66$ ft. per sec. at 63° 26′ with the X axis.

2. The position of a body is given by $x = \sin t$, in which x is given in feet and t is given in seconds. Find the velocity when $t = 1$ second and when $t = 2$ seconds.

Ans. $v_1 = 0.540$ ft. per sec.; $v_2 = -0.416$ ft. per sec.

3. The position of a body is given by $x = r \cos \omega t$, $y = r \sin \omega t$, in which r is a constant length expressed in feet and ω is a constant number. Find the magnitude and direction of the resultant velocity for any given value of the time

Ans. $v_x = -r\omega \sin \omega t$; $v_y = r\omega \cos \omega t$; $v = r\omega$, at an angle with the X axis whose tangent is equal to $-\text{cotan } \omega t$.

Problem 3 gives the motion of a body moving in a circle of radius r with uniform speed $r\omega$. In Fig. 236, I, the body is at B and is moving in a counter-clockwise direction. The radius from the center to B makes an angle ωt with the horizontal line OA. Fig. 236, II, is the vector velocity triangle. The horizontal component v_x is toward the left, while the vertical component is

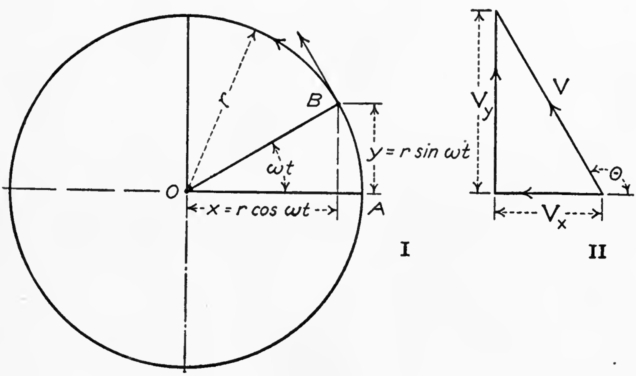

FIG. 236.

upward. The resultant velocity v makes an angle θ with the horizontal toward the right.

$$\text{Tan } \theta = -\text{cotan } \omega t,$$

which shows that the resultant velocity is normal to the line OB. The line which is tangent at B in Fig. 236, I, gives the direction of the resultant velocity.

160. Acceleration.—The rate of change of velocity is called the *acceleration.* If a body is moving east with a velocity of 10 feet per second, and 2 seconds later is moving east with a velocity of 40 feet per second, the change in velocity is 30 feet per second east. This change takes place in 2 seconds. The rate of change is $30 \div 2 = 15$ feet per second per second.

Problems

1. If the acceleration is 4 feet per second per second east and the body has an initial velocity of 12 feet per second east, what will be its velocity 5 seconds later?

Ans. $v = 32$ ft. per sec. east.

2. If the acceleration is 4 feet per second per second east and the body has an initial velocity of 12 feet per second west, what will be its velocity 4 seconds later? *Ans.* $v = 8$ ft. per sec. east.

3. A body is moving north 20 feet per second. Two minutes later it is moving north 92 feet per second. Find its acceleration.

Ans. 36 ft. per sec. per min.; 0.6 ft. per sec. per sec.

The expression for the component of the acceleration in the direction of the X axis is

$$a_x = \frac{v_2 - v_1}{t_2 - t_1}, \qquad \text{Formula XVIII}$$

in which v_1 is the component of the velocity in the direction of the X axis at the time t_1, v_2 is the component at the time t_2, and a_x is the acceleration.

Problems

4. A car is moving on a track with increasing velocity. By means of a electro-magnet actuated by a current through a break-circuit chronometer, a mark is made on the track every 2 seconds. At 0 seconds the mark is 16 feet from a reference point. At 2 seconds the mark is 46 feet from the reference point, and at 4 seconds the mark is 100 feet from the reference point. Find the acceleration.

The average velocity for the first 2-second interval is 15 feet per second. For the next interval, the average velocity is 27 feet per second. Assuming that the average velocity is the velocity at the middle of the interval, the the velocity at 1 second is 15 feet per second and the velocity at 3 seconds is 27 feet per second. The acceleration is

$$a = \frac{27 - 15}{3 - 1} = 6 \text{ ft. per sec. per sec.}$$

The problem may be arranged in this way:

Time	Displacement	Velocity	Acceleration
0	16		
1	....	15	
2	46		6 ft. per sec. per sec.
3	....	27	
4	100		

5. In an experiment similar to Problem 4, the readings were 1 second, 24 feet; 3 seconds, 60 feet; 5 seconds, 140 feet. Find the acceleration.

6. In an experiment similar to Problem 4, the readings were 1 second, 20 feet; 3 seconds, 54 feet; 7 seconds, 218 feet. Find the acceleration.

Ans. 8 ft. per sec. per sec.

7. In Problem 4, suppose that the acceleration is constant. Find the position at the 6th second.

8. In Problem 1 of Art. 157, find the velocity at 4 seconds and at 5 seconds by means of intervals of 0.2 second; then find the average acceleration for the one-second interval.

Ans. Average a = 16 ft. per sec. per sec.

9. In problem 3 of Art. 157, find the approximate velocity at 4 seconds and at 6 seconds by means of 2-second intervals, and then find the average acceleration for the 2-second interval.

Ans. a = 60 ft. per sec. per sec.

10. A cannon ball acquires a velocity of 1800 feet per second in 0.004 second. Find its acceleration. *Ans.* a = 450,000 ft. per sec. per sec.

11. A baseball with a speed of 100 feet per second is stopped in 0.001 second. Find its acceleration.

12. The velocity of a body is changed from 80 feet per second east to 40 feet per second west in 3 seconds. Find its acceleration.

Ans. a = 40 ft. per sec per sec. west.

161. Acceleration as a Derivative.—When the increment of time and the corresponding increment of velocity are very small, Formula XVIII becomes

$$a_x = \frac{dv_x}{dt}. \tag{1}$$

Since $v_x = \frac{dx}{dt}$, Equation (1) reduces to

$$a_x = \frac{d^2x}{dt^2} \tag{2}$$

The acceleration along the other coordinate axes is given by similar derivatives.

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2};$$
$$a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}; \qquad \text{Formulas XIX}$$
$$a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}.$$

In some works of mathematical physics the components of velocity and acceleration are written thus: The component of the velocity parallel to the X axis is $\dot{x}$ and the component of the acceleration is $\ddot{x}$. A derivative with respect to time is written with a single dot over the variable. The second derivative with respect to time is written two dots over the variable.

Problems

1. The position of a body is given by $x = kt^2$. When the displacement varies as the square of the time, show that the acceleration is constant.

Ans. $a = 2k$.

2. Find the acceleration in Problem 1 of Art. 157 by means of the second derivative. *Ans.* $a = 16$ ft. per sec. per sec.

3. Find the acceleration in Problem 3 of Art. 157 when $t = 5$ seconds.
Ans. $a = 12t = 60$ ft. per sec. per sec.

The answer of Problem 3 is the same as the answer of Problem 9 of the preceding article for the average acceleration for the interval from 4 seconds to 6 seconds. While the average velocity differs considerably from the velocity at the middle of the interval, the acceleration obtained is the correct value for the middle of its assumed interval.

4. The position of a body is given by the equation $y = e^t$. Find the velocity and acceleration when $t = 2$ seconds.
Ans. $v_y = e^2 = 7.39$; $a_y = 7.39$.

5. The position of a body is given by the equation $x = 10 \sin 4t$. Find the velocity and acceleration when $t = 1$ second and when $t = 2$ seconds.

162. Acceleration as a Vector.—Since velocity is a vector and acceleration is velocity divided by time (which is not a vector), it follows that acceleration is a vector quantity. Accelerations may be resolved into components or combined into resultants in the same way as forces or other vectors.

Problems

1. The position of a body is given by the equations $x = 2t + 3t^2$, $y = t^3$. Find the coordinates of its position and the direction and magnitude of its velocity and acceleration when $t = 4$ seconds. Plot the curve of the path of the body, the curve of its velocity, and the curve of its acceleration.

2. The position of a body is given by $x = 4 \cos t$, $y = 4 \sin t$. Find the components of the velocity and acceleration when $t = 0$, when $t = \frac{\pi}{2}$ and when $t = \pi$.

Ans. $t = 0$, $v_x = 0$; $v_y = 4$; $a_x = -4$; $a_y = 0$.
$t = \frac{\pi}{2}$, $v_x = -4$; $v_y = 0$; $a_x = 0$; $a_y = -4$.
$t = \pi$, $v_x = 0$; $v_y = -4$; $a_x = 4$; $a_y = 0$.

Problem 2 is that of a body which moves in the circumference of a circle of 4-foot radius with a speed of 4 feet per second. When $t = 0$, the body is at position A of Fig. 237. The motion is directly upward. The vertical component of the velocity is the entire velocity of 4 feet per second, while the horizontal component is zero. Since the vertical component is not changing, $a_y = 0$. The *direction* of the velocity as a vector is continually changing. When the body is at a small distance above A, its velocity has a component toward the left. There is an acceleration toward the left to give it this change of velocity. This acceleration is 4 ft. per sec. per sec. At B, where $t = \frac{\pi}{2}$, the entire velocity is horizontal toward the left. It is represented by $v_x = -4$, $v_y = 0$. The acceleration is 4 ft. per sec. per sec. downward.

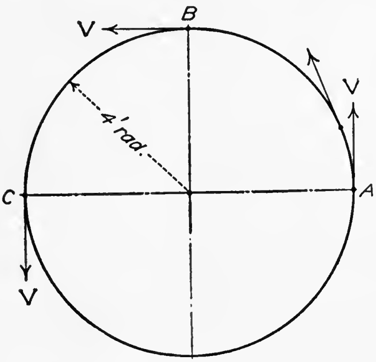

FIG. 237.

The results of Problem 2 show that the acceleration is constant in magnitude and is directed toward the center, while the velocity is constant in magnitude and is directed along the tangent.

Problem

3. The position of a body is given by the equations, $x = r \cos \omega t$, $y = r \sin \omega t$, in which r and ω are constants. Find the magnitude and direction of the acceleration for any given time.

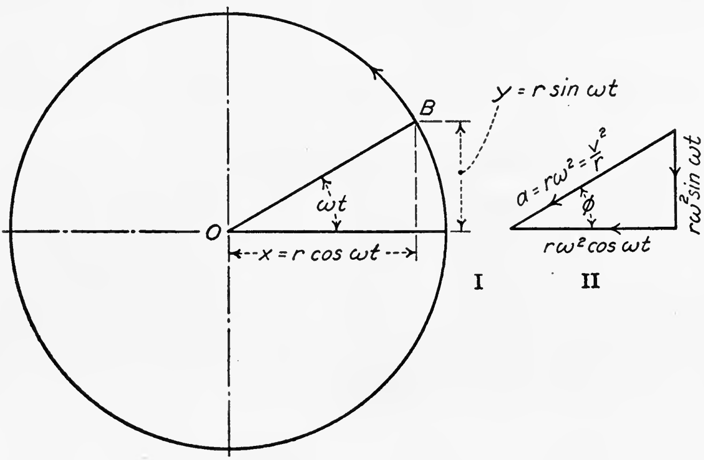

FIG. 238.

Ans. $a_x = -r\omega^2 \cos \omega t$; $a_y = -r\omega^2 \sin \omega t$; resultant acceleration $= r\omega^2$.

Problem 3 gives the motion of a body which moves in a circle of radius r with a linear velocity of $r\omega$. (Problem 3, Art. 159). If the linear velocity $r\omega = v$,

$$\text{acceleration} = r\omega^2 = \frac{r^2\omega^2}{r} = \frac{v^2}{r}. \qquad \text{Formula XX}$$

Figure 238, II, is the vector triangle of acceleration for Problem 3 when the body is at position *B* of Fig. 238, I. The horizontal component is negative, and is, therefore, directed toward the left. The vertical component is negative, and is, therefore, directed downward. If ϕ is the angle which the resultant acceleration makes with the horizontal,

$$\tan \phi = \frac{-r\omega^2 \sin \omega t}{-r\omega^2 \cos \omega t} = \tan \omega t.$$

When the body is at *B* of Fig. 238, I, the acceleration is in the direction *BO*. When a body is moving in a circle with constant speed, its acceleration is constant in magnitude and is directed toward the center of the circle. The acceleration is due to the *change of direction of the velocity* and not to change of magnitude.

If a body is moving in any curved path with uniform speed, the acceleration is directed toward the center of curvature of the path. The magnitude of the acceleration is $\frac{v^2}{r}$, in which r is the radius of curvature of the path at the given position.

4. A body is moving east 20 feet per second. Two seconds later it is moving north 20 feet per second. Find the average acceleration during the 2-second interval.

Figure 239 is the vector diagram of velocities. The increment of velocity is $\Delta \mathbf{v}$ which must be added to $\mathbf{v}_1$ to get $\mathbf{v}_2$. The vector equation is

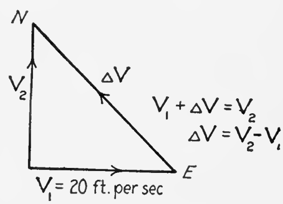

FIG. 239.

$$\mathbf{v}_1 + \Delta\mathbf{v} = \mathbf{v}_2;\ \mathbf{v}_2 - \mathbf{v}_1 = \Delta\mathbf{v}.$$

By trigonometry,

$$\Delta\mathbf{v} = 28.28 \text{ ft. per sec. northwest.}$$

$$a = \frac{28.28}{2} = 14.14 \text{ ft. per sec. per sec. directed northwest.}$$

5. A base ball is pitched horizontally with a velocity of 100 feet per second. It leaves the bat with a velocity of 120 feet per second at an angle of 30 degrees with the horizontal and passes directly over the pitchers box. What is the change of velocity? If the ball is in contact with the bat for 0.02 second, what is the acceleration?

Ans. a = 10,628 ft. per sec. per sec. at 16° 24′ with the horizontal.

6. A ball is thrown against a wall with a velocity of 100 feet per second and rebounds in the opposite direction with a velocity of 80 feet per second, what is the acceleration?

7. Solve Problem 5 if the ball passes directly over first base.

8. A body moves in the circumference of a circle of radius r feet per second with a velocity of v feet per second. Find the direction and magnitude of the acceleration as a vector.

Figure 240, I, shows the circle of Problem 8. When the body is at A, the velocity is $\mathbf{v}_1$; when the body is at B, the velocity is $\mathbf{v}_2$. The velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ have equal magnitudes. Their directions, on the other hand, are different. Since $\mathbf{v}_1$ is normal to the radius OA and $\mathbf{v}_2$ is normal to the radius OB, the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ is equal to the angle AOB. Figure 240, II, is the vector diagram of the velocities. The change in velocity is the vector Δv. If this change takes place in time Δt, the acceleration is $\frac{\Delta v}{\Delta t}$. The direction of the acceleration is the direction of Δv. From Fig. 240, II, $\Delta v = v\Delta\theta$, in which v is the magnitude of the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. The distance from A to B of Fig. 240, I, measured along the arc of the circle, is $r\Delta\theta$. The body moving with a speed v passes over this distance in time Δt.

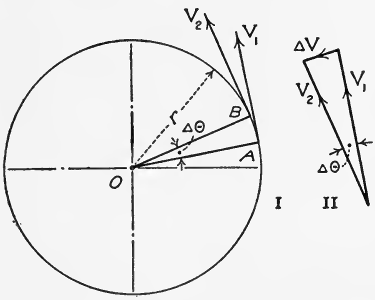

FIG. 240.

$$r\Delta\theta = v\Delta t; \quad \Delta t = \frac{r\Delta\theta}{v};$$

$$a = \frac{\Delta v}{\Delta t} = v\Delta\theta \div \frac{r\Delta\theta}{v} = \frac{v^2}{r}. \qquad \text{Formula XX}$$

The direction of Δv is perpendicular to the bisector of the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ and is, therefore, parallel to the bisector of the angle AOB. As $\Delta\theta$ is made smaller and smaller, the direction of the acceleration is always parallel to this bisector. When the angle becomes infinitesimal, it is evident that the acceleration is directed toward the center of curvature. The methods of Problem 3 differ from those of Problem 8. The results, however, are identical.

163. Dimensional Equations.—A dimensional equation is an equation which shows with what powers the fundamental quantities of mass, length, and time appear in the expression for a physical quantity. In these equations, mass is represented by M, time is represented by T, and length is represented by L. Some dimensional equations are:

$$\text{Area} = L^2;$$
$$\text{Volume} = L^3;$$

$$\text{Density} = \frac{M}{L^3} = ML^{-3};$$

$$\text{Velocity} = \frac{L}{T} = LT^{-1};$$

$$\text{Acceleration} = \frac{L}{T^2} = LT^{-2}.$$

Negative exponents are generally used instead of fractional forms.

Dimensional equations have two important uses. These are:

1. A dimensional equation may be used to determine whether a literal expression for a quantity is possibly correct.

2. In transferring from one system of units to another, a dimensional equation indicates the required power of the ratio of the units of the two systems.

The dimensions of an area are $A = L^2$. If an area in square yards is transferred to square feet, the ratio of the number of units of length in feet to the number of units in yards is 3. Since L in the dimensional equation is squared, it follows that the number of square feet in a given area is 3^2 times as great as the number of square yards. This is an example of the second use of a dimensional equation.

If a, b, and c are lengths, an area may be expressed by a^2, b^2, c^2, ab, ac, bc, $c\sqrt{a^2 + b^2}$, etc. If an expression which is known to be an area has any of these forms, or any other form which is equivalent to the square of a length, the expression represents a area. Such expressions as abc, $\frac{a^2 + b^2}{c}$, $\sqrt{abc}$, etc, can not represent areas. If such an expression appears in a quantity which is known to be an area, it will be recognized as incorrect. These are examples of the first use of a dimensional equation.

Only quantities having the same dimensions may be added or subtracted. Volume can not be added to area nor velocity to density. If a, b, and c are lengths, and m is a mass, $ab + b^2 + a\sqrt{bc}$ and $mc + m\sqrt{a^2 + c^2}$ are possible additions.

Problems

1. The dimension of the trigonometric functions is zero, since each function is the ratio of two lengths. If a, b, c, and s are lengths, is it possible to represent a sine by $\frac{b}{\sqrt{a^2 + c^2}}$, by $\frac{ab}{c^2}$, by $\sqrt{\frac{(s-b)(s-c)}{bc}}$, or by $\frac{s^2 + bc}{a}$? What is the nature of the quantity expressed by $\sqrt{s(s-a)(s-b)(s-c)}$? What is the quantity expressed by $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$?

2. If w represents mass, q represents time, and a, b, and c represent lengths, state whether the following are possible physical quantities, and give the meaning of those which are possible:

$$\frac{w}{abc}, \quad \frac{\sqrt{ab}}{q}, \quad \frac{a^2 + b^2}{w}, \quad \frac{a^2 + b^2 + c^2}{q^2}.$$

3. What is the coefficient required to reduce area in square feet to area in square inches and to area in square yards? *Ans.* 12^2; $(\frac{1}{3})^2$.

4. What is the factor required to reduce pounds per square inch to grams per square centimeter?

5. How is velocity in feet per second reduced to velocity in miles per hour?

6. How is acceleration in feet per minute per minute reduced to acceleration in feet per second per second?

164. Summary.—Displacement is the amount of change of position. Displacement is a vector.

The displacement per unit of time is the velocity. Velocity is a vector. The component of the velocity parallel to the X axis is

$$v_x = \frac{x_2 - x_1}{t_2 - t_1} = \frac{dx}{dt}.$$

The dimensional equation of velocity is LT^{-1}.

Change of velocity per unit of time is acceleration. Since velocity is a vector, the change of velocity may involve change in direction, change in magnitude, or change in both direction and magnitude. Acceleration is a vector. The component of acceleration parallel to the X axis is

$$a_x = \frac{v_2 - v_1}{t_2 - t_1} = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}.$$

When a body is moving in a circle of radius r with a velocity of v units per unit of time, the acceleration toward the center due to the change of direction is

$$a = \frac{v^2}{r}.$$

The dimensional equation of acceleration is LT^{-2}.

A dimensional equation is useful in transferring from one system of units to another and in checking literal equations.

CHAPTER XVII

FORCE AND MOTION

165. Force and Acceleration.—If the forces which act on a body are in equilibrium, the body either remains at rest or moves in a straight line with constant speed. There is no change of motion. Newton's First Law states: A body at rest remains at rest and a body in motion continues to move uniformly in a straight line unless constrained by some external force to change that condition. If a horizontal sheet of ice were perfectly frictionless, a body sliding on it would continue to move in one direction with constant speed for an indefinite distance. The attraction of the earth downward on the body and the reaction of the ice upward are in equilibrium and do not affect the motion. If there is some friction between the body and the ice, the velocity gradually decreases until the body finally comes to rest. The force of friction opposite the direction of the motion produces a negative acceleration in the body. The ice sheet may be so inclined to the horizontal that the component of the weight down the incline exactly equals the friction. If, then, the body is given an initial velocity down the plane, it will continue to move in that direction with constant speed. In that case, the forces are in equilibrium and cause no change in motion. If the slope of the plane is increased so that the component of the weight down the plane exceeds the force of friction in the opposite direction, the velocity of the body down the plane will continually increase. The effective force is down the plane in the direction of the initial velocity and the acceleration is positive.

The acceleration of a body is proportional to the resultant effective force which acts on it from other bodies and is inversely proportional to its mass. If an unbalanced force of 1 pound accelerates a given mass 5 feet per second per second, a force of 2 pounds will accelerate the mass 10 feet per second per second and a force of 3 pounds will accelerate it 15 feet per second per second. If a given force accelerates a mass of 1 pound 16 feet per second per second, that force will accelerate a mass of 2 pounds 8 feet per second per second and will accelerate a mass 8 pounds 2 feet per second per second. These statements are based on experiments and on deductions from observed facts. The acceleration is in the direction of the resultant force. Newton's Second Law states: Change of motion is proportional to the force applied and takes place in the direction in which the force acts. Expressed algebraically,

$$\text{acceleration} = \frac{\text{a constant} \times \text{effective force}}{\text{mass of the body}}$$

$$a = \frac{kP}{m}; \tag{1}$$

$$P = \frac{ma}{k}, \qquad \text{Formula XXI}$$

in which P is the unbalanced force, a is the acceleration in the direction of the force, m is the mass, and k is a constant. The value of the constant k depends upon the units of force, mass, length, and time which are employed. These units may be so chosen that k is equal to unity.

The most common case of uniformly accelerated motion is that of a freely falling body. If the density of the body is sufficiently great so that the resistance of the air may be neglected without appreciable error (or if the body falls in a vacuum), the only force acting on the body is the attraction of the earth. The effective force is the weight of the body. Experiments show that the acceleration of a freely falling body is a little over 32 feet per second per second. This figure is called the acceleration of gravity and is represented in algebraic equations by the letter g.

If a freely falling body has a mass of one pound, the effective force (which is equal to its weight) is one pound. A force of one pound gives to a mass of one pound an acceleration of g feet per second per second. If the mass of a freely falling body is m, its weight is m pounds. Substituting in Equation (1)

$$a = g = \frac{km}{m} = k. \tag{2}$$

When the effective force is expressed in pounds, the mass is expressed in pounds, and the acceleration is given in feet per second per second, the constant k of Equation (1) and Formula XXI is equal to g. In general, when the unit of force is taken as the weight of the unit of mass, as is done in practically all engineering and commercial work, the constant k is equal to g and Formula XXI becomes

$$P = \frac{ma}{g}. \qquad \text{Formula XXII}$$

If the acceleration is given in feet per second per second, the "standard" value of g is 32.174. Formula XXII becomes effective force in pounds =

$$\frac{\text{mass in pounds} \times \text{accel. in ft. per sec. per sec.}}{32.174}$$

The standard value of g is the acceleration of a freely falling body at the sea level at 45 degrees latitude. It is sometimes written g_0. The average g for the British Isles is about 32.2. This value is largely used in English books. The average g for The United States is less than 32.174. The value of 32.16 is in common use.

Formula XXII may be written

$$P = m\frac{a}{g}. \qquad (3)$$

This form of the equation states that the effective force is equal to the product of the mass multiplied by the ratio of its acceleration to the acceleration of gravity. If a 10-pound shot is accelerated 16 feet per second per second, the effective force is $10 \times \frac{16}{32} = 5$ pounds.

Examples

Solve these examples without writing, using 32 feet per second per second as the value of g.

1. A mass of 24 pounds has an acceleration of 8 feet per second per second. What is the effective force?

2. A mass of 48 pounds moving with a velocity of 80 feet per second is brought to rest in 5 seconds. What is the acceleration and what is the effective force?

3. A force of 20 pounds acts on a 10-pound mass for 2 seconds. What is the acceleration and what velocity will the body acquire if initially at rest?

4. A force of 12 pounds acting for 3 seconds gives a body a velocity of 24 feet per second. The body was at rest at the beginning of the interval. What is its mass in pounds?

Problems

Use $g = 32.174$

1. A mass of 40 pounds moving at the rate of 60 feet per second on a horizontal plane is brought to rest in 6 seconds by the friction of the plane. Find the retarding force in pounds and the coefficient of friction.

Ans. $P = 12.43$ lb.

2. A body is moving on a horizontal plane with a velocity of 50 feet per second. If the coefficient of friction is 0.3, what is the negative acceleration and in what time will the body come to rest?

$a = 9.65$ ft. per sec. per sec.

3. A body is placed on a smooth inclined plane, which makes an angle of 20 degrees with the horizontal. What is the effective force in pounds, and what is the acceleration down the plane?

Ans. $P = 0.342$ m.; $a = 11.00$ ft. per sec. per sec.

4. A body weighing 30 pounds is placed on an inclined plane, which makes an angle of 25 degrees with the horizontal. The coefficient of friction is 0.12. Find the effective force down the plane and find the acceleration.

Ans. $P = 9.42$ lb.; $a = 10.10$ ft. per sec. per sec.

5. The projectile from a 12-inch naval gun weighs 1000 pounds and acquires a velocity of 2800 feet per second in 0.035 second. If the pressure is constant throughout this interval, find the total pressure and the pressure per square inch. *Ans.* Total pressure = 2,486,000 lb.

166. Constant Force.—When the resultant force is constant, the acceleration is constant and the velocity and displacement are easily calculated. If a is the acceleration, the change of velocity in t seconds is at feet per second. If v_0 is the velocity at the beginning of the time t, the velocity at the end of the interval is

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a}t. \tag{1}$$

Unless the initial velocity and the acceleration are in the same direction, Equation (1) must be treated as a vector equation.

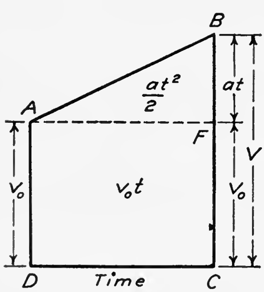

FIG. 241.

In Fig. 241, time is plotted as abscissas and velocity as ordinates. The velocity curve is the straight line AB with slope equal to the acceleration.

Displacement is the product of velocity multiplied by time. When the velocity varies, the displacement is the product of the average velocity multiplied by the time interval. When the acceleration is constant, as in Fig. 241, the average velocity is the velocity at the middle of the time interval, or one-half the sum of the initial and final velocities.

$$\text{Average velocity} = \frac{v_0 + v}{2} = \frac{2v_0 + at}{2} = v_0 + \frac{at}{2}. \tag{2}$$

The last form of Equation (2) states that the average velocity, when the acceleration is constant, is the initial velocity plus one-half of the increase during the interval t.

Examples

Solve without writing

1. The velocity of a body changes from 20 feet per second east to 40 feet per second east in 5 seconds. What is the average velocity and what is the distance traversed in 5 seconds?

Ans. Average velocity = 30 ft. per sec.; displacement = $30 \times 5 = 150$ ft.

2. A body moving with a velocity of 40 feet per second is brought to rest in 4 seconds. How far does it go? *Ans.* $20 \times 4 = 80$ ft.

3. A body moving 40 feet per second is brought to rest in 100 feet. What is the time interval, and what is the acceleration?

The average velocity is 20 feet per second. The time is $100 \div 20 = 5$ seconds. The change in velocity is 40 feet per second and takes place in 5 seconds. The acceleration is $-40 \div 5 = -8$ ft. per sec. per sec.

4. In Problem 3, the mass is 60 pounds. What is the retarding force in pounds?

5. A body has its velocity changed from 20 feet per second east to 50 feet per second east while moving 140 feet. Find its acceleration.

6. A body has its velocity changed from 40 feet per second east to 20 feet per second west in 6 seconds. What is its average velocity for the 6 seconds? How far is it from the starting point at the end of the 6 seconds? How far does it actually travel in the 6 seconds?

The displacement is the product of the average velocity multiplied by the time. If s represents the displacement in any direction,

$$s = \left(v_0 + \frac{at}{2}\right)t = v_0t + \frac{at^2}{2}. \qquad (3)$$

The first term of the second member of Equation (3) gives the displacement which results from the initial velocity. The second term gives the additional displacement which results from the acceleration, or the displacement which would exist if the body had no initial velocity.

Equation (3) may be represented by the area of the diagram of Fig. 241. The rectangle *DAFC* represents v_0t; the triangle *ABF* represents $\frac{at^2}{2}$.

If the initial velocity and the acceleration are not in the same direction, Equation (3) must be regarded as the sum of two vectors.

In solving problems, it is recommended that the student make use of the average velocity as in the preceding examples and employ Equation (3) as little as possible.

Problems

1. A mass of 40 pounds is moving east with a velocity of 50 feet per second. After an interval of 4 seconds, the body is 120 feet farther east. Find the final velocity, the acceleration, and the effective force. Solve by means of average velocity and the definition of acceleration. Check by Equations (1) and (3).

2. A 40-pound mass has a velocity of 60 feet per second east. It is subjected to a force of 10 pounds north. What will be its position and velocity at the end of 3 seconds?

Ans. Displacement = 183.6 ft. north 78° 37′ east of the initial point; v = 64.68 ft. per sec. north 68° 05′ east.

3. A body is moving east with a velocity of 50 feet per second. After 6 seconds, it is moving west with a velocity of 10 feet per second. How far does it move during the 6 seconds, and how far is it from the initial point at the end of 6 seconds? Draw a diagram similar to Fig. 241 to give its velocity and position.

4. A body starts up a 20-degree inclined plane with a velocity of 60 feet per second. The coefficient of friction is 0.2. What is the negative acceleration when the body is going up the plane? In what time will it come to rest and what will be its distance from the initial point? What is the acceleration when the body is coming down the plane? How long will it take to come down and what will be the final velocity at the bottom?

Ans. Distance = 105.57 ft.

5. Solve Problem 4 if the coefficient of friction is 0.4.

6. A 20-pound mass moving with a velocity of 100 feet per second is stopped in 2 feet. Find the acceleration and the effective force.

Ans. a = 2500 ft. per sec. per sec.; P = 777 lb.

7. A base ball weighing 5. 5 oz. and moving with a speed of 100 feet per second is stopped in 1 foot. Find the average force.

Ans. P = 53.4 lb.

8. Solve Problem 7 if the ball is stopped in 1 inch.

9. A cannon ball weighing 1000 pounds acquires a velocity of 2800 feet per second while moving 50 feet. Find the acceleration and the effective force on the projectile. *Ans.* P = 2,436,000 lb.

10. The projectile of Problem 9 strikes a steel plate and is stopped in 2 feet. Find the force.

11. An automobile is traveling on a level street with a velocity of 30 feet per second. The coefficient of starting friction is 0.35. Half the weight comes on the rear wheels. In what time can the car be brought to rest and how far does it go, if the brakes are set so that the wheels do not skid?

Ans. t = 5.33 sec.; s = 80 ft.

12. Solve Problem 11 if the coefficient of sliding friction is 0.25 and the brakes are set too tight so that the wheels skid.

13. Solve Problem 11 if the speed is only 15 feet per second.

14. Solve Problem 11 for a speed of 30 miles per hour, 15 miles per hour, and 10 miles per hour.

15. An elevator starts from rest and is accelerated upward 6 feet per second per second for 3 seconds. It then moves upward with constant speed for 5 seconds. For the last 26 feet the acceleration is downward. What is the total distance? What is the effective force on a man weighing 160 pounds during each stage of the ascent? What is the pressure between his shoes and the floor? If the man stands on a platform scale, what is his apparent weight during each stage?

16. What is the direction of the acceleration of an elevator at the bottom when it is starting up, and when it is coming to a stop at the bottom? What is the direction of acceleration at the top when it is starting down, and when it is coming to a stop at the top?

167. Integration Methods.—Equations (1) and (3) of the preceding article were derived by algebraic methods. These are the better methods for constant acceleration, as they give greater prominence to the physical ideas than do the methods of calculus. With variable acceleration, calculus is necessary. In order to become familiar with the application of calculus, it is advisable to apply it to constant acceleration and check the results by means of the equations of the preceding article.

$$\frac{dv}{dt} = a. \tag{1}$$

Multiplying by dt and integrating

$$v = at + C_1, \tag{2}$$

in which C_1 is an integration constant. If v_0 is the velocity when $t = 0$, $C_1 = v_0$ and Equation (1) becomes

$$v = v_0 + at, \tag{3}$$

which is Equation (1) of the preceding article.

Substituting $v = \dfrac{dx}{dt}$ in Equation (3).

$$\frac{dx}{dt} = v_0 + at, \tag{4}$$

$$x = v_0 t + \frac{at^2}{2} + C_2. \tag{5}$$

If space is measured from the position where $t = 0$, then $C_2 = 0$, and

$$x = v_0 t + \frac{at^2}{2}. \tag{6}$$

If space is measured from some other point such that $x = x_0$ when $t = 0$, then $C_2 = x_0$ and

$$x = x_0 + v_0 t + \frac{at^2}{2}. \tag{7}$$

Problem

The acceleration of a body is proportional to the time and is 8 feet per second per second when $t = 2$ seconds. The velocity is 10 feet per second in the direction of the acceleration when $t = 1$ second. By integration, find the equations of velocity and displacement in terms of the time and calculate the velocity and displacement when $t = 5$ seconds.

Ans. $v = 58.0$ ft. per second; $s = 123.3$ ft. from the position when $t = 0$.

168. Connected Bodies.—When the *equilibrium* of connected bodies is considered, the entire system is first taken as a free body for the determination of the external reactions. The system is then divided into parts, and each portion is treated as a free body in equilibrium under the action of the external forces and of the internal forces from the adjacent portions of the system. This method is used in the calculation of the forces in trusses and other structures. In a similar manner, Formula XXII may be applied to an entire system or to any part of the system.

Example

Figure 242 shows a 20-pound mass on a smooth horizontal plane. A cord attached to the mass runs horizontally over a smooth pulley and supports a 12-pound mass. Assuming that the cord and the pulley are weightless and that the plane and pulley are frictionless, find the acceleration and the tension in the cord.

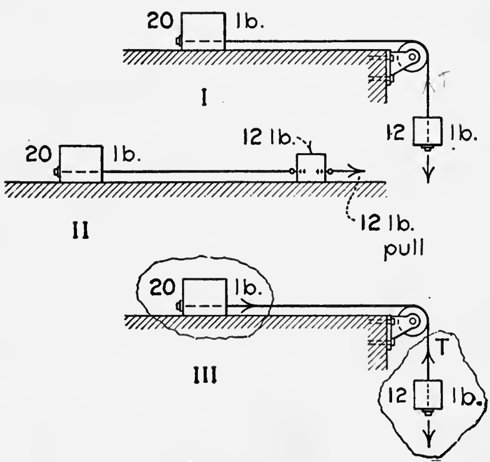

FIG. 242.

The two masses and the connecting cord form a system. The weight of the 20-pound mass and the reaction of the plane are in equilibrium. The unbalanced effective force is the pull of the earth on the 12-pound mass. The pulley merely changes the direction of the cord. Mechanically the system is equivalent to Fig. 242, II, in which both masses are on the same horizontal plane and a horizontal force of 12 pounds is applied to the 12-pound mass

Considering the system as a whole, the effective force is 12 pounds and the mass which must be accelerated is 32 pounds. Substituting in Formula XXII,

$$12 = \frac{32a}{g};$$

$$a = \frac{12 \times 32.174}{32} = 12.065 \text{ ft. per sec. per sec.}$$

To determine the tension in the cord, which is an internal force between portions of the system, the 12-pound mass may now be treated as a free body. The forces which act on this free body are the pull of the earth downward and the tension of the cord upward. The effective force is $12 - T$ pounds, Fig. 242, III. The mass which is accelerated by this force is 12 pounds. The acceleration is now known to be 12.045 or $\frac{12g}{32}$ feet per second per second.

$$12 - T = \frac{12 \times 12.065}{32.174} \text{ or } \frac{12 \times \frac{12g}{32}}{g} = 4.5 \text{ lb.};$$

$$T = 12 - 4.5 = 7.5 \text{ lb.}$$

As a check, the 20-pound mass may be considered as the free body. The effective force is T pounds toward the right and the mass is 20 pounds.

$$T = \frac{20 \times a}{g} = \frac{20 \times \frac{12g}{32}}{g} = 7.5 \text{ lb.}$$

Problems

1. Solve the example above if the coefficient of friction between the 20-pound mass and the plane is 0.1.

Ans. $a = \frac{(12 - 2)}{32}g = 10.054$ ft. per sec. per sec.; $T = 8.25$ lb.

2. A 40-pound mass is placed on a plane which makes an angle of 20 degrees with the horizontal. The mass is attached to a cord which runs up parallel to the plane, passes over a smooth pulley, and supports a 30-pound mass on the free end. Find the acceleration and the tension on the cord if the plane is smooth. Check.

Ans. $P = 16.32$ lb.; $a = \frac{16.32g}{70} = 7.501$ ft. per sec. per sec.; $T = 23.02$ lb.

3. Solve Problem 2 if the coefficient of friction between the plane and the 40-pound mass is 0.1.

Ans. $a = 5.776$ ft. per sec. per sec.; $T = 24.62$ lb.

4. What velocity will the bodies of Problem 2 acquire in 3 seconds and what distance will they travel in 3 seconds if they start from rest?

5. A mass of 20 pounds is on a horizontal plane. It is attached to a cord which runs horizontally over a smooth weightless pulley and supports a mass of 16 pounds. The system starts from rest and travels 24 feet in the first 2 seconds. Find the coefficient of friction between the 20-pound mass and the horizontal plane. *Ans.* $f = 0.129$.

6. Figure 243 shows a cord which runs over a pulley and supports a mass of 6 pounds on one end and a mass of 4 pounds on the other. If the pulley is weightless and frictionless and if the cord is weightless, what velocity will the system acquire in 4 seconds after starting from rest? What distance will it travel in the 4 seconds? What is the tension in the cord? What is the total load on the pulley. If this load is less than 10 pounds explain the discrepancy?

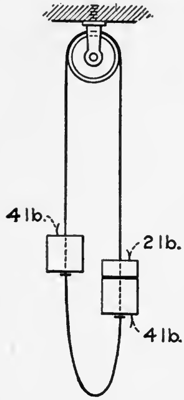

Fig. 243.

Figure 243 represents the Atwood machine, which is considerably used to demonstrate the laws of acceleration and to determine g approximately. The machine is subject to errors which result from the friction and from the mass of the pulley. These errors may be determined experimentally and corrections made. Error caused by the varying length of cord on the two sides may be avoided by having the cord continuous as is shown in the figure. The total mass of the cord must be added to that of the two suspended bodies to get m of Formula XXII.

Problems

7. A 20-pound mass is placed on a plane which makes an angle of 20 degrees with the horizontal. The mass is attached to a cord which runs up the plane, passes over a smooth weightless pulley, and supports a mass of 12 pounds on the free end. Starting from rest, the system moves 18 feet during the first 3 seconds. Find the coefficient of friction between the plane and the 20-pound mass.

Ans. $F = 1.172$ lb.; $f = 0.063$.

8. In Fig. 244, the movable pulley and the mass W together weigh 160 pounds. The mass P is 100 pounds. Find the acceleration of each body and the tension in the rope if the mass of the fixed pulley and the rope is negligible and there is no friction. *Ans.* $T = 85.71$ lb.

Suggestion: In this problem it is best to write a separate equation for each of the two bodies and combine these equations to solve for T and a.

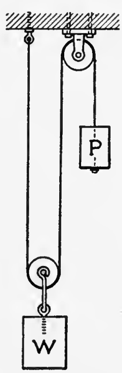

Fig. 244.

169. Velocity and Displacement.—The equations of Art. 167 give the displacement in terms of the initial velocity, the acceleration, and the time. It is sometimes desirable to express the displacement in terms of the initial and final velocities and the acceleration, and to eliminate the time.

$$s = \frac{v_0 + v}{2}t. \tag{1}$$

From the definition of acceleration,

$$t = \frac{v - v_0}{a}; \qquad (2)$$

$$s = \frac{v^2 - v_0}{2a}; \qquad (3)$$

$$v^2 - v_0^2 = 2\ as. \qquad (4)$$

When the initial velocity is zero, Equation (4) becomes

$$v^2 = 2as, \qquad \text{Formula XXIII}$$

If the final velocity is zero,

$$v_0^2 = 2as, \qquad (5)$$

in which a is negative.

Problems

1. The acceleration of a given body is 8 feet per second per second. What is its velocity after moving 36 feet from the position of rest? Solve by Formula XXIII. Check by means of the time.

2. The velocity of a body changes from 4 feet per second to 12 feet per second while it travels 32 feet. Find the acceleration. Solve by one of the equations above and check by means of the average velocity and the time.

3. The acceleration of a body is 12 feet per second per second. How far must it travel in order that the velocity may change from 4 feet per second to 16 feet per second in the same direction?

4. The answer to Problem 3 is the same whether the initial velocities are in opposite directions or in the same direction. What is the meaning of s when the velocities are in opposite directions?

170. Energy.—A force of P pounds acts on a mass of m pounds which is moving with a velocity v feet per second in the direction of the force. During a time interval dt, the displacement is $ds = vdt$ and the work done by the force of P pounds is

$$dU = Pds = Pvdt. \qquad (1)$$

The force P accelerates the body of mass m. The value of the force in terms of the mass and the acceleration is

$$P = \frac{ma}{g} = \frac{m}{g}\frac{dv}{dt}. \qquad (2)$$

Substituting in Equation (1) and integrating,

$$dU = P\ vdt = \frac{mvdv}{g}; \qquad (3)$$

$$U = \frac{mv^2}{2g}. \qquad \text{Formula XXIV}$$

If the body starts from rest, the limits of Formula XXIV are

0 and v. The formula, therefore, gives the entire work done in increasing the velocity from zero to v. It is not necessary that the force and acceleration should be constant in order that these equations may be valid.

Formula XXIV is the expression for the *kinetic energy* of the body. This energy depends upon the mass and the velocity. It is immaterial how the velocity is obtained. Since v^2 is positive whether v is positive or negative, the kinetic energy is independent of the direction of the motion. Kinetic energy is, therefore, a scalar quantity.

In Formula XXIV, if the mass is given in pounds and the velocity in feet per second, g is 32.174 and the energy is given in foot pounds.

$$\text{Kinetic energy in foot pounds} = \frac{\text{mass in pounds } (v \text{ in feet per sec.})^2}{2 \times 32.174}.$$

Problems

1. A car weighing 2400 pounds is moving with a velocity of 20 feet per second. Find its kinetic energy in foot-pounds. If the car is brought to rest while moving 80 feet, what is the average retarding force?

Ans. Kinetic energy = 14,919 foot-pounds; force = 186.5 lb.

2. Solve Problem 1 if the velocity is 40 feet per second.

3. The car of Problem 1 has its velocity changed from 40 feet per second to 20 feet per second while it runs 100 feet. Find the force required.

4. What is the kinetic energy of a cannon ball weighing 1000 pounds and moving with a velocity of 2800 feet per second. If this projectile is fired from a gun which is 50 feet in length, what is the average force?

5. The projectile of Problem 4 strikes a wall and is stopped in 10 feet. Find the average pressure.

Problems of accelerated motion (whether positive or negative acceleration) in which the *distance* and change in velocity are given are best solved by equating the change of kinetic energy with the work done on the body when the acceleration is positive or with the work done by the body when the acceleration is negative. Problems in which the *time* and the change in velocity are given are best solved by means of the acceleration and Formula XXII.

Example

A mass of 40 pounds has its velocity changed from 60 feet per second to 20 feet per second while it goes 200 feet. Find the force required.

$$\text{Change in kinetic energy} = 40\frac{3600-400}{2 \times 32.174} = 1989 = 200\ P;\ P = 9.95 \text{ lb.}$$

To solve this example by means of the acceleration,

$$\text{average } v = \frac{80 + 20}{2} = 40 \text{ ft. per sec.;}$$

$$t = \frac{200}{40} = 5 \text{ sec.;}$$

$$a = \frac{20 - 60}{5} = -8 \text{ ft. per sec. per sec.;}$$

$$P = \frac{40 \times 8}{32.174} = 9.95 \text{ lb.}$$

It is evident that the method of kinetic energy is the shorter for a problem in which the distance is given.

171. Potential Energy.—In Art. 121, *potential energy* was defined as the energy of position. If a 12-pound mass is lifted a vertical distance of 10 feet, work done on the mass is 120 foot-pounds and the mass can do 120 foot-pounds of work as it returns to its original position. The change of potential energy is 120 foot-pounds.

If no energy is lost by change into heat or by work on other bodies, the sum of the potential and kinetic energy of a body or system of bodies remains constant. This statement, which is based on experiments, is called the Law of Conservation of Energy. Energy may be transformed from one kind to another but can not be destroyed. Many problems of mechanics may be solved by means of this principle.

Example

Solve Problem 8 of Art. 168 by means of the energy relations. Assume that the mass of 100 pounds moves downward a distance of s feet. The work is 100 s foot-pounds and the potential energy of the system is reduced that amount. At the same time the mass of 160 pounds is raised a distance of $\frac{s}{2}$ feet and the potential of the system is increased 80 foot-pounds. The total change of potential energy is 20 s foot-pounds. Equating the change of potential energy with the change of kinetic energy,

$$20\,s = \frac{100\,v^2}{2g} + \frac{160\left(\frac{v}{2}\right)^2}{2g};$$

$$v^2 = \frac{2gs}{7}.$$

From Formula XXIII,

$$v^2 = 2as;$$

$$a = \frac{g}{7} = 4.595 \text{ ft. per sec. per sec.}$$

Problems

1. A body slides 50 feet down a smooth inclined plane which makes an angle of 25 degrees with the horizontal. What is its velocity at the end of its descent? *Ans.* $21.13\ m = \frac{mv^2}{2g}$; $v = 36.8$ ft. per sec.

2. The body of Problem 1 weighs 20 pounds and the friction is equivalent to a force of 2 pounds. Find the final velocity.

Ans. 322.60 ft.-lb. $= \frac{20\ v^2}{2g}$; $v = 32.2$ ft. per sec.

3. Solve Problem 1 if the coefficient of friction is 0.2.

4. A block and tackle has three ropes which support the weight (Fig. 192). The movable pulley and its load weigh 240 pounds. A mass of 120 pounds is hung on the free end of the rope. Find the velocity of the free end of the rope when the load is lifted 20 feet. Find the acceleration and the tension on the rope.

5. A mass of 3 pounds is placed on one end of the cord of an Atwood machine and a mass of 2 pounds is placed on the other end. The efficiency of the pulley is 95 per cent. What is the velocity after moving 10 feet? What is the tension in the cord?

172. Motion Due to Gravity.—When a body is not supported, its weight is the effective force, and its acceleration downward is equal to g. If displacement and velocity downward are taken as positive, then the force of gravity and the acceleration are positive. In the equations of the preceding articles the acceleration a becomes g and

$$v = v_0 + gt, \tag{1}$$

$$h = v_0 t + \frac{gt^2}{2}, \tag{2}$$

$$v^2 - v_0^2 = 2gh, \tag{3}$$

in which h is the vertical distance and is positive downward. When the initial velocity is zero, Equation (3) becomes

$$v^2 = 2gh \qquad \text{Formula XXIII}$$

This form of Formula XXIII is important. It gives the velocity which a body will acquire in falling from a given height, the height to which a body will rise with a given initial velocity upward, and the velocity of flow of a liquid from an orifice under a given head.

When the initial velocity is upward, it is often desirable to regard velocity and displacement upward as positive. The acceleration of gravity is then taken as a negative quantity.

Example

A body is thrown upward with a velocity of 80 feet per second. What will be its velocity when it is 20 feet above the starting point?

Using Equation (3) with h positive and g negative,

$$v^2 = 6400 - 1286.96;$$
$$v = 71.5 \text{ feet per second.}$$

Problems

1. In the example above there are two solutions for the velocity. Explain.

2. A body is thrown vertically upward with a velocity of 100 feet per second. How high will it rise? In what time will it return to the starting point? *Ans.* $h = 155.4$ ft.; time of ascent and return $= 6.216$ sec.

Solve Problems 3, 4, *and* 5 *without writing. Use* $g = 32$ *and employ the average value of the velocity when convenient.*

3. A body is thrown upward with a velocity of 80 feet per second. How long will it rise? What is the average velocity upward? How high will it go?

4. A body is thrown upward with a velocity of 80 feet per second. What is the average velocity during the first second? How far does it go during the first second?

5. A body is thrown upward with a velocity of 40 feet per second. What is its average velocity during the first second? Where will it be at the end of the first second? What is its average velocity during the first 2 seconds? Where will it be at the end of 2 seconds? How far will it travel during the first 2 seconds?

6. Derive Formula XXIII by equating the potential and kinetic energy.

7. A body is thrown upward with a velocity of 120 feet per second. At what point will the velocity be 60 feet per second? At what point will the velocity be 150 feet per second?

8. In what time will a body fall 100 feet if it has an initial velocity of 60 feet per second downward?

8. Derive Equation (3) by means of kinetic and potential energy.

173. Projectiles.—A body which is thrown into the air and continues its motion under the action of no external force except its weight and the resistance of the air is called a projectile. For the present, consideration will be given to comparatively heavy projectiles moving at relatively low velocities. The resistance of the air may be neglected for such projectiles and the only force is that due to gravity. The problems of the preceding article are examples of projectiles with initial velocity vertical.

The only effect of gravity is to change the vertical component of the velocity. The horizontal component remains constant. The velocity at any instant is the resultant of the horizontal and vertical components.

Figure 245 shows positions of a body A which has been thrown horizontally. At the same time, a second body B has been allowed to fall. Body A moves over equal horizontal distances during each interval of time, and falls vertically through the same distances as a body which drops directly downward. The first body A may be thrown from a spring gun. The second body B may be supported by an electromagnet the circuit of which is broken when the projectile leaves the gun. If their paths do not intersect, both bodies will strike the horizontal floor at exactly the same instant. If their paths intersect, the bodies will collide.

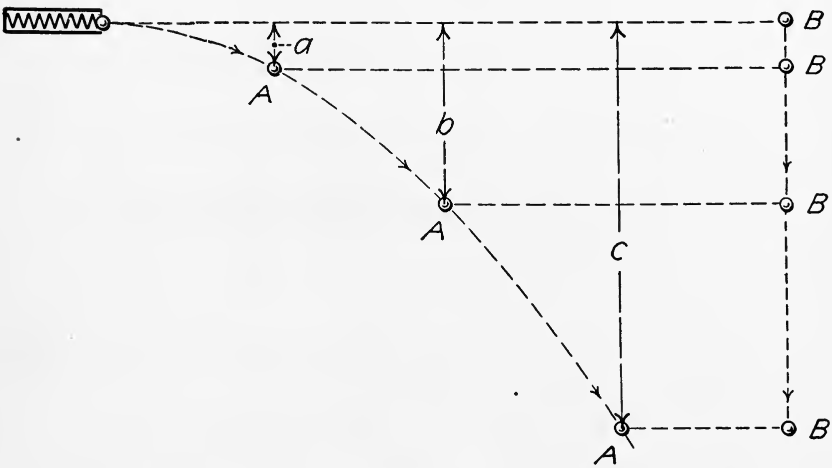

Fig. 245.

In Fig. ,246, the first body is thrown at an angle with the horizontal and the second body is dropped at the same time from a point which is in the line of the initial velocity of the first body. The distance which the projectile falls from the line of its initial velocity is exactly the same as in the case when the initial velocity is horizontal or zero. The projectile may be regarded as moving with constant speed along the line of its initial velocity and falling from that line with constant acceleration. If the path of the projectile and the falling body intersect, they will collide. If the paths do not intersect, they will strike a horizontal floor simultaneously.[1]

[1] This interesting experiment was suggested by Professor F. E. Kester of the University of Kansas.

Problems

1. A body is projected horizontally with a speed of 40 feet per second. Where will it be at the end of one second, and what will be the direction and magnitude of its resultant velocity?

Ans. $x = 40$ ft.; $y = -16.09$ ft.; $v = 51.4$ ft. per sec. at an angle of 38° 49′ with the horizontal.

2. A train is running over a trestle with a velocity of 40 feet per second. A lump of coal falls from the tender and strikes the ground 30 feet below. Find its resultant velocity by means of the energy equations.

Ans. $v = 59.4$ ft. per sec.

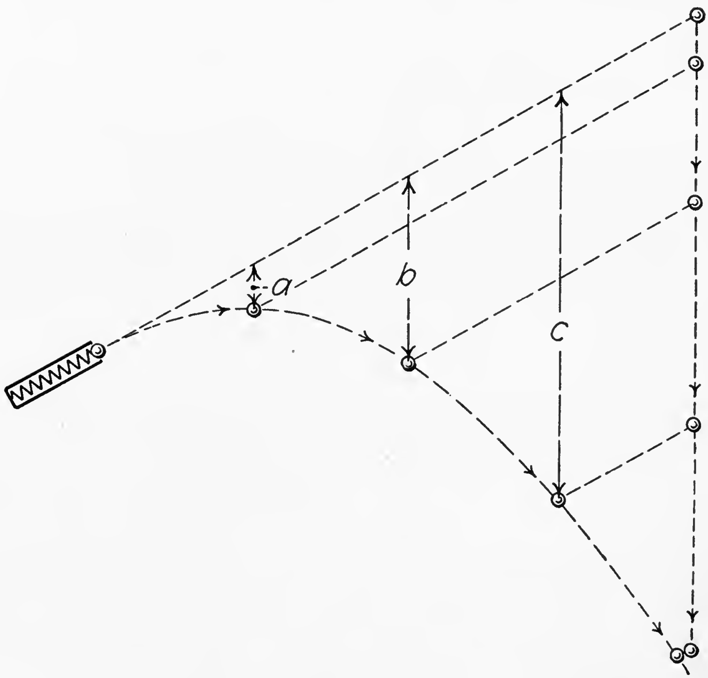

FIG. 246.

3. A train is running over a trestle with a velocity of 40 feet per second. A lump of coal is thrown horizontally at right angles to the direction of the track with a velocity of 30 feet per second. Find the direction and magnitude of the velocity after 2 seconds.

Ans. $v = 81.5$ ft. per sec. at 52° 09′ with the horizontal.

4. A body is thrown upward at an angle of 30 degrees with the horizontal with a velocity of 140 feet per second. Find the horizontal component and the vertical component of its velocity after 2 seconds.

Ans. $v_x = 121$; 24 ft. per sec. $v_y = 5.65$ ft. per sec. upward.

5. In Problem 4, how long will the body continue to rise? How high will it go? Solve for the height by means of the average velocity.

Ans. 2.176 sec.; 76.16 ft.

6. In what time will the body of Problem 4 return to the ground? How far from the starting point will it strike? *Ans.* $x = 527.6$.

If a projectile is thrown upward at an angle α with the horizontal with a velocity of v feet per second, the components of its velocity after t seconds are

$$v_x = v_0 \cos \alpha, \tag{1}$$

$$v_y = v_0 \sin \alpha - gt. \tag{2}$$

The time required to reach the highest point in the path is found by equating the vertical component to zero. This method is equivalent to treating the vertical component as the velocity of a body moving vertically upward.

To find the position at any time t,

$$x = (v_0 \cos \alpha)\ t, \tag{3}$$

$$y = (v_0 \sin \alpha)t - \frac{gt^2}{2}. \tag{4}$$

The first term of Equation (4) gives the distance the body would rise with a constant velocity $v_0 \sin \alpha$. The last term gives the distance it would fall. The difference of the terms gives the distance above the horizontal line through the starting point.

Problems

7. Eliminate t from Equations (3) and (4) and derive the equation of the path of the projectile in terms of x and y.

8. Derive the expression for the horizontal range of the projectile.

$$Ans.\quad x = \frac{v_0^2 \sin 2\alpha}{g}.$$

9. What is the range of a projectile if the initial velocity is 160 feet per second at an angle of 35 degrees with the horizontal?

Ans. $x = 747.7$. ft.

10. Show that the range is a maximum if the initial velocity makes an angle of 45 degrees with the horizontal.

11. If the resistance of the air may be neglected, what is the maximum distance which a basball may be thrown with an initial velocity of 100 feet per second? *Ans.* 310.8 ft.

12. A 16-pound shot leaves an athlete's hand at a distance of 7 feet above the ground with a velocity of 36 feet per second. How far will it go, if thrown at 45 degrees with the horizontal? *Ans.* 46.36 ft.

13. The shot of Problem 12 moves a distance of 7 feet before leaving the athlete's hand. What is the average resultant force. Solve by work and energy. *Ans.* 57.3 lb.

174. Summary.—Newton's Laws of Motion are:

1. Every body continues in a state of rest or of uniform motion in a straight line, unless impelled by some external force to change that condition.

2. Change of momentum [Momentum is the product of the mass and velocity] is proportional to the applied force and takes place in the direction in which the force acts.

3. For every action there is an equal and opposite reaction.

A force which changes the motion of a body must be exerted by some outside body. The attraction of the Earth for a projectile changes the motion of the projectile. The attraction of the projectile for the Earth changes the motion of the Earth. Regarded as a single system the motion of the center of mass of the projectile and the Earth is not changed.

The term "applied force" means the resultant, unbalanced, effective force.

Newton's Second Law is expressed algebraically by the formula,

$$P = \frac{ma}{g}, \qquad \text{Formula XVII}$$

in which the force P is expressed in terms of the weight of unit mass. When the time is expressed in seconds and the distance in feet so that the acceleration is in feet per second per second, then $g = 32.174$. If the distance is given in centimeters, the value of g is 981.

Acceleration is the rate of change of velocity.

$$a = \frac{v_2 - v_1}{t_2 - t_1} = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

$$v_x = \frac{x_2 - x_1}{t_2 - t_1} = \frac{dx}{dt}.$$

The displacement is the product of the average velocity multiplied by the time. When the acceleration is constant, the displacement may be represented by the area of a trapezoid whose base is the time and whose terminal ordinates are the initial and final velocities. The average velocity, when the acceleration is constant, is the velocity at the middle of the time interval.

Kinetic energy is the energy of motion. Its equation is

$$U = \frac{mv^2}{2g}. \qquad \text{Formula XXIV}$$

Potential energy is the energy of position. The sum of the potential and kinetic energies of a system remains constant unless work is done by outside bodies.

Displacement, velocity, and acceleration are vectors. Work and energy are not vectors.

When a body is moving in a circle with uniform speed v, the acceleration toward the center due to the change in direction is

$$a = \frac{v^2}{r}.$$

If a projectile has a low velocity and considerable density, so that the influence of the air resistance is negligible, the horizontal component of the velocity will remain unchanged throughout the flight and the vertical component will be the initial vertical component minus the change due to gravity.

Formula XXII applies to an entire connected system or to any part of such system.

175. Miscellaneous Problems

1. A mass of 20 pounds slides down a plane, which is 60 feet long and makes an angle of 20 degrees with the horizontal. After leaving the inclined plane, it slides on a horizontal plane. The coefficient of friction of the inclined plane is 0.2 and the coefficient of friction of the horizontal plane is 0.25. How far will the body go on the horizontal plane? Solve by work and energy. What is the acceleration on each plane?

Ans. 36.97 ft.; 4.96 ft./sec²; −8.04 ft./sec.²

2. A man standing on the ground throws a ball weighing 1 pound with a horizontal velocity of 60 feet per second. What energy does he give to the ball?

Ans. $U = \frac{1800}{g} = 55.95$ ft.-lb.

3. The man of Problem 2 stands on a car which is moving at the rate of 40 feet per second and throws the 1-pound ball with a velocity of 60 feet per second relative to the car in the direction the car is moving. How much energy does he give to the ball? He exerts the same force as when he was standing on the ground. Explain the difference.

Ans. Energy change = 130.53 ft.-lb.

4. The man of Problem 2 stands on a car which is moving with a speed of 40 feet per second and throws a ball weighing 1 pound with a velocity of 60 feet per second in a horizontal direction at right angles to the direction of the car's motion. How much energy does he impart to the ball? What is the total energy after the ball has fallen 20 feet? What is the direction of the motion after the ball has fallen 20 feet?

Ans. 55.95 ft.-lb.; 100.76 ft.-lb. 26° 27′ with the horizontal in a vertical plane which makes an angle 56° 19′ with the direction of motion of the car.

5. A rope is thrown over a cylinder whose axis is horizontal. A 4-pound mass is hung on one end of the rope and a 1-pound mass is hung on the other end. The coefficient of friction is 0.2. Find the tension on each end of the rope and the acceleration.

6. A 40-pound mass has its velocity changed during 4 seconds from 120 feet per second east to 40 feet per second east. Find the force required.

7. A 40-pound mass has its velocity changed from 120 feet per second east to 40 feet per second east while it goes 240 feet. Find the acceleration by means of the average velocity and find the force required to change the motion. Solve also for the force by means of work and energy.

8. A 40-pound mass has its velocity changed from 120 feet per second east to 40 feet per second west during an interval of 5 seconds. How far does it travel during the 5 seconds and how far is the final position from the initial position? Draw the diagram of velocity and time, and check the distances by means of the areas.

9. A mass of 60 pounds has its velocity changed from 80 feet per second east to 60 feet per second north in an interval of 3 seconds. Find the direction and magnitude of the resultant force.

10. A body slides down an inclined plane and out upon a horizontal plane as in Problem 1. If the coefficient of friction is the same for both planes, and if no additional energy is lost when the direction of the motion is changed, show that the final position will be the same no matter what is the angle of the inclined plane. Show also that the horizontal distance of the final position from the initial position is equal to the initial height multiplied by the cotangent of the angle of friction.

CHAPTER XVIII

SYSTEMS OF UNITS

176. Gravitational System.—The system of units used in the preceding chapter may be called the *gravitational system*. The unit force in this system is the weight of unit mass. With English units the unit of mass is the pound and the unit of force is the weight of one pound mass,—the pound mass is the unit of mass and the pound weight is the unit of force. With metric units the unit of mass is the gram or the kilogram and the unit of force is the weight of a gram mass or of a kilogram mass. These are the ordinary units in every day use. They have the apparent disadvantage that the weight of unit mass varies slightly with latitude and altitude. The *standard unit* of force is the weight of the unit of mass at 45 degrees latitude at the sea level.

In the gravitational system, the constant in the equation which expresses the relation of force, mass and acceleration is g.

$$P = \frac{ma}{g}; \qquad \text{Formula XXII}$$

$$\text{kinetic energy} = \frac{mv^2}{2g}. \qquad \text{Formula XXIV}$$

177. Absolute Systems.—In the *absolute systems*, the constant in the equation of force, mass, and acceleration is unity. The equation reads

$$P = ma \qquad \text{Formula XXV}$$

If $m = 1$ and $a = 1$ in Formula XXV, then the formula defines P as the force which gives unit acceleration to unit mass. The equation gives a definition of the unit of force which is everywhere constant.

Since force is the product of mass multiplied by acceleration, and since the dimensional equation of acceleration is LT^{-2}, it follows that the dimensional equation of force is MLT^{-2}.

The expression for kinetic energy in an absolute system is

$$U = \frac{mv^2}{2}.$$

The dimensional equation of kinetic energy is ML^2T^{-2}. The dimensional equation of work is the same as that of kinetic energy.

178. The Centimeter-Gram-Second System.—An absolute system in which the unit of mass is the gram and the unit of acceleration is the centimeter per second per second is called the centimeter-gram-second system. The name is abbreviated to the C.G.S. system.

In this system, the unit of force is the *dyne*. The dyne is that force which gives to a gram mass an acceleration of 1 centimeter per second per second. The weight of a gram mass gives to a gram an acceleration of g centimeters per second per second. The value of g varies slightly. For most places it is between 980 and 981. The weight of a gram is about 981 dynes. In the C.G.S. system, weight $= mg$; in the gravitational system, weight $= m$.

The unit of energy in the C.G.S system is called the *erg*. An erg is the work done by a force of one dyne when the displacement is 1 centimeter. Ergs are reduced to gram-centimeters of work by dividing by 981.

The C.G.S system is used by physicists. It has the advantage of *apparent* simplicity and it gives a unit of force which does not change with locality. The commercial electrical units were originally based on the C.G.S system. The *official* definitions now used do not, however, depend directly upon this system.

Problems

1. A mass of 20 grams has its velocity changed from 60 centimeters per second to 20 centimeters per second in 4 seconds. Find the effective force in dynes and the change in kinetic energy.

Ans. $P = 200$ dynes; change in energy $= 32{,}000$ ergs.

2. Calculate the displacement in Problem 1 by means of the average velocity and compute the work done by the force.

3. A mass of 50 grams is placed on one side of an Atwood machine and a mass of 30 grams on the other Find the acceleration and the tension in the cord, if g is 980.6 at that locality.

4. What is the kinetic energy in ergs of a mass of 2 kilograms which is moving with a velocity of 4 meters per second. *Ans.* $U = 16 \times 10^7$ ergs.

5. A mass of 20 kilograms is moving on a horizontal plane with a velocity of 5 meters per second. It is brought to rest in 10 meters by the friction of the plane. Find the coefficient of friction. *Ans.* $f = 0.127$.

6. How many dynes are equal to a standard pound force?

Ans. 444,820 dynes = 1 standard pound.

7. Reduce a standard foot-pound to ergs.

Ans. $13,558 \times 10^3$ ergs = 1 standard foot-pound.

8. How many dynes are equal to the weight of one gram at 45 degrees latitude at the sea level? *Ans.* 980.66 dynes.

179. The Foot-Pound-Second System.—The Foot-Pound-Second System is an absolute system in which the pound is the unit of mass and the foot per second per second is the unit of acceleration. The unit of force is called the *poundal.* A poundal is that force which gives to a pound mass an acceleration of 1 foot per second per second. Since the weight of a pound mass gives to a pound mass an acceleration of g feet per second per second, it is evident that a pound force is equal to g poundals force. A standard pound force is 32.174 poundals. A poundal is a little less than the weight of one-half ounce. The unit of work and energy in this system is the foot-poundal.

Problems

1. A mass of 60 pounds is moving with a velocity of 20 feet per second. Find the force in poundals which will stop it in a space of 20 feet.

Ans. $P = 600$ poundals.

2. A car weighing 2400 pounds is moving with a velocity of 30 feet per second. Find its kinetic energy in foot-poundals.

Ans. 108×10^4 foot-poundals.

3. A 40-pound mass on a 20-degree inclined plane is attached to a rope which runs up the plane, passes over a smooth pulley, and supports a mass of 24 pounds. Find the acceleration and the tension on the rope in poundals.

In problem 3, and in all problems in which a force depends upon the weight of some body, it is necessary to know the local value of g in order to get the true results in poundals. If an assumed value of g is used, the results will be in error in the ratio of the assumed value of g to the true local value. When the Formulas of Art 165 are employed with the standard value of g, the acceleration of Problem 3 will be in error in the ratio of the standard g to the local g. The tension, on the other hand, will be correct in local pounds. As far as the absolute accuracy of the results are concerned, one system has no advantage over the other.

Forces are usually measured by means of the local weight of a copy of a standard mass. When it is desired to calibrate a spring balance, standard "weights" are hung on it. (Methods for calibrating a spring directly in absolute units are given in Chapter

XIX.) Spring balances are generally graduated to read in pounds and not in poundals.

In poundals, $P = ma$; Formula XXV

in pounds, $P = \frac{poundals}{g} = \frac{ma}{g}$. Formula XXII

For the student who has begun with the absolute systems, Formula XXII may be regarded as obtained from Formula XXV by division by g.

180. The Engineer's Unit Mass.—A force of one pound gives to a mass of one pound an acceleration of 32.17 feet per second, per second. A force of one pound gives to a mass of 32.17 pounds an acceleration of 1 foot per second per second. A mass of 32.17 pounds is called the *engineer's* unit of mass. It seems to be assumed that it is desirable to express Newton's Second Law in the form of

$$\text{Force} = \text{mass} \times \text{acceleration} \tag{1}$$

When the force in Equation (1) is expressed in pounds, it is necessary to have this large unit of mass. This unit of mass until recently had no name except "engineer's unit," or simply the "unit of mass." Professor E. R. Maurer has suggested the name of *gee-pound,* since the unit is g times as large as the pound unit. (Some British writers call it the slugg.) To find the number of *gee-pounds* in a given mass, divide the mass in pounds by g. Some engineering writers express the number of pounds mass by W. The number of gee-pounds is $\frac{W}{g}$. If M is the mass in gee-pounds, $M = \frac{W}{g}$. Newton's Second Law is written,

$$\text{Force in pounds} = \frac{W}{g}a. \tag{2}$$

Since W is really the mass in pounds (though it is often called the weight)

$$\frac{Wa}{g} = \frac{ma}{g},$$

and Equation (2) is equivalent to Formula XXII. The difference between Equation (2) as used by some writers and Formula XXII lies in the fact that $\frac{W}{g}$ is defined as being the mass of the body in a new unit, while in Formula XXII, W or m represents the mass in pounds, and $\frac{a}{g}$ is the ratio of the acceleration of the body to the acceleration of gravity, or g is regarded as a constant occurring in the formula.

It is sometimes stated that W in this equation is the *weight* of the mass on a standard spring balance. A body may weigh 332 pounds at one locality and 321.5 pounds at another. If g at the first locality is 32.2 and at the second locality is 32.15, the mass is found to be 10 units in each case. The formula $P = Ma$ gives the correct accelerating force in standard pounds, and the formula $U = \frac{Mv^2}{2}$ gives the correct energy in standard foot-pounds. Unfortunately, the use of such a standard spring balance assumes that the true value of g is known. The same errors, therefore, come in as with the use of other methods. Moreover, such standard spring balances do not exist, and would not be sufficiently accurate for refined experiments if they were made. When it is necessary to measure forces with an accuracy greater than the variation of gravity, beam balances are employed and the value of g is determined by vibration experiments.

The formula $P = \frac{Wa}{g}$ may be written,

$$\frac{P}{W} = \frac{a}{g}. \tag{3}$$

In this equation W is regarded as a force. The effective force is to the weight of the body as the acceleration produced by the force is to the acceleration produced by the weight of the body. Equation (3) may be regarded as the simplest form of the equation of acceleration. It does not involve any unusual unit, such as the poundal force or the gee-pound mass. If only linear acceleration were to be considered, Equation (3) would be the best form to use. On account of the necessity of applying the formulas to angular acceleration and to problems of kinetic energy, it is advisable to emphasize the idea of the mass. Formula XXII is, therefore, best adapted for general use.

181. Summary.—Systems of units depend upon the value of the constant k in the equation $P = \frac{ma}{k}$. The constant is unity in the absolute systems and the equations for acceleration and kinetic energy are written

$$P = ma; \qquad \text{Formula XXV}$$

$$U = \frac{mv^2}{2}$$

In these systems, the units of mass, length, and time are arbitrarily defined and the unit of force is derived from Formula XXV. The results are changed into gravitational units by dividing by g. In the C.G.S. system, the dyne is the unit of force. A dyne is the force which gives to a gram mass an acceleration of 1 centimeter per second per second. In the foot-pound-second system, the poundal is the unit of force. A poundal is the force which gives to a pound mass an acceleration of 1 foot per second. One pound force is equal to 32.174 poundals. One gram force is equal to 981 dynes.

In the so-called Engineer's System, the units of time and length are the same as in the other systems. The unit of force is taken arbitrarily and the unit of mass is defined by Formula XXV. Standard units of force and mass, as required by this system, do not exist. Physicists use the pound as the unit of mass and define the unit of force by Formula XXV. Engineers, also, use the pound as the unit of mass and define the unit of force as the weight of a pound mass at the standard position. Since nothing is gained by the use of the so-called Engineer's Unit of mass, it might well be dropped from technical literature.

Many books have the formulas,

$$\text{Force in pounds} = \frac{Wa}{g};$$

$$\text{Kinetic energy in foot-pounds} = \frac{Wv^2}{2g}.$$

These are identical with the formulas of the preceding chapter except that W is used instead of m for the mass in pounds. The student is advised to think of g as a mere coefficient occurring in the formula on account of the units employed, and not to consider $\frac{W}{g}$ as representing the mass.

The absolute systems have their uses in some scientific work. In problems where forces depend upon the weight of some bodies, it is necessary to know the local value of g. In these systems, weight = mg. While these systems eliminate g from the formula for acceleration, they insert g into many problems of statics.

Engineers are advised to learn Formulas XXII and XXIV as given in the preceding chapter. These formulas are apparently more complex than those of the other systems. It is easier, however, for the engineer who is not working daily with such ideas to remember (or look up) an equation with an additional letter in it than to remember to divide by g to transform to practical units.

The systems of units are given in Table 1 below.

TABLE I.—SYSTEMS OF UNITS

System	Force	Mass	Length	Weight	Formulas
Gravitational.	Pound	Pound	Foot	m	$P = \frac{ma}{g}$; $U = \frac{mv^2}{2g}$
C.G.S.	*Dyne*	Gram	Cm.	mg	$P = ma$; $U = \frac{mv^2}{2}$
Ft. Lb.; S. . . .	*Poundal*	Pound	Ft.	mg	$P = ma$; $U = \frac{mv^2}{2}$
Engineer's. . . .	Pound	*Gee-pound*	Ft.	mg	$P = ma$; $U = \frac{mv^2}{2}$

CHAPTER XIX

FORCE WHICH VARIES AS THE DISPLACEMENT

182. The Force of a Spring.—The deformation of a spring or other elastic body is proportional to the force. Figure 247, I, shows a helical spring which is not loaded. A load of 1 pound stretches the spring a definite amount d as shown in Fig. 247, II. A load of 2 pounds would produce an elongation of $2d$. The forces on the spring are downward and the elongation is downward. The force with which the spring acts on the loads is upward opposite to the deformation.

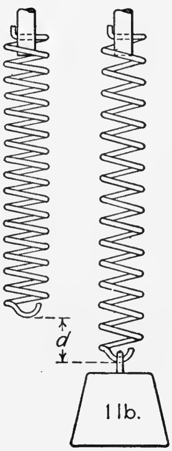

Fig. 247.

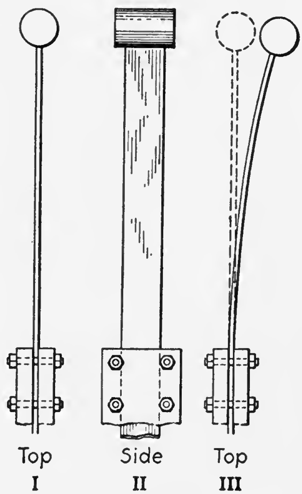

Fig. 248.

When a body is attached to a spring, there is some position at which it remains at rest. In this position, either the force of the spring is zero, or the resultant of the force of the spring and the other forces is zero. Figure 248 represents a body on a flat strap spring. The plane of the paper is supposed to be horizontal in Fig. 248, I. The weight of the body tends to bend the spring in a vertical direction but not in a horizontal direction. Equilibrium exists, therefore, when the spring viewed from above, appears straight. If the body is displaced toward the right, as shown in Fig. 248, III, the spring tends to move it backward toward the left. If it is displaced toward the left from the position of equilibrium, the spring tends to move it toward the right.

If the spring in Fig. 248 exerts a force of K pounds when it is deformed a distance of 1 foot, it will exert a force of $2K$ pounds when it is deformed a distance of 2 feet and a force of Kx pounds when it is deformed a distance of x feet. The force acts in a direction opposite to the displacement. The direction and magnitude of the *force exerted by the spring on the body* is given mathematically by the equation.

$$P = -Kx. \qquad (1)$$

In Equation (1) K is the force of the spring when the displacement is unity, x is the displacement from the position of equilibrium, and P is the force of the spring when the displacement is x. The negative sign indicates that the direction of the force which the spring exerts on the body is opposite the direction of the displacement.

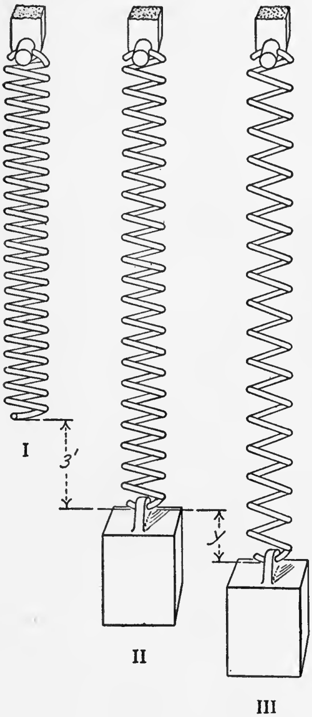

FIG. 249.

Equation (1) applies also when a body is subjected to the force of a spring and a constant force. The displacement is measured from the position of equilibrium at which the force of the spring is equal and opposite to the constant force. Figure 249 shows a vertical helical spring. It is assumed that a load of 24 pounds stretches this spring a distance of 3 feet, as shown in Fig. 249, II. The constant K is 8 pounds. In Fig. 249, II, the mass of 24 pounds is subjected to a downward pull of 24 pounds which is due to gravity, and to an upward pull of 24 pounds which is due to the reaction of the spring. If the 24-pound mass is now displaced an additional foot downward, the elongation of the spring becomes 4 feet and the force in it becomes 32 pounds. The force of gravity remains 24 pounds. The resultant force on the 24-pound mass is an upward force of 8 pounds. If the 24-pound mass is displaced a distance y from the position of equilibrium, the total elongation of the spring is $3 + y$ feet, and the upward pull of the spring becomes $24 + 8y$ pounds. The downward pull of gravity is still 24 pounds. The resultant force on the body, when the displacement is $-y$ feet, is equal to $8y$ pounds. If the 24-pound mass is displaced a distance y upward from the position of equilibrium, the total elongation of the spring is $3 - y$ feet and the pull of the spring is $24 - 8y$, provided y is not more than 3 feet. The downward pull of the earth is still 24 pounds so that the resultant force on the body is $-8y$ pounds when the displacement is $+y$ feet from the position of equilibrium.

If the origin of coördinates is taken at the position of equilibrium under the action of the constant force and the reaction of the spring, the expression for the resultant force on the body is $P = -Kx$ or $P = -Ky$.

If the displacement in Fig. 249 is greater than 3 feet, the spring will come into compression. If the force required to compress the spring unit distance is the same as the force required to stretch it unit distance, Equation (1) will remain valid. A long helical spring such as is shown in Fig. 249 will buckle in compression so that the force required to compress it unit distance is not the same as the force which stretches it unit distance. With such a spring Equation (1) will not hold when it is shortened below its natural length.

Problems

1. A spring board is deflected 5 inches downward by a load of 100 pounds on the free end. What is the value of K? *Ans.* $K = 240$ lb.

2. What is the deflection of the spring board of Problem 1 when a boy weighing 120 pounds stands on the end? *Ans.* 0.5 ft.

3. In Problem 2, what is the resultant force on the boy when the spring board is deflected 8 inches below the position at which it stands when not loaded. *Ans.* 40 lb. upward.

4. In Problem 2, what is the resultant force on the boy when the spring is deflected 3 inches downward from the position of equilibrium under no load? *Ans.* 60 lb. downward.

5. Solve Problem 4 if the deflection is 1 inch upward from the position of equilibrium under no load and the boy is not fastened to the spring board. *Ans.* $P = -120$ lb.

6. A body is placed on a horizontal plane and attached to one end of a spring. The natural length of the spring is 3 feet and a pull of 12 pounds stretches it 6 inches. A cord attached to the body runs horizontally over a smooth pulley and supports a mass of 60 pounds, as shown in Fig. 250, II. What is the zero position of the right end of the spring when the system is in equilibrium? When the spring is stretched to a length of 7 feet, what is the resultant force on the body?

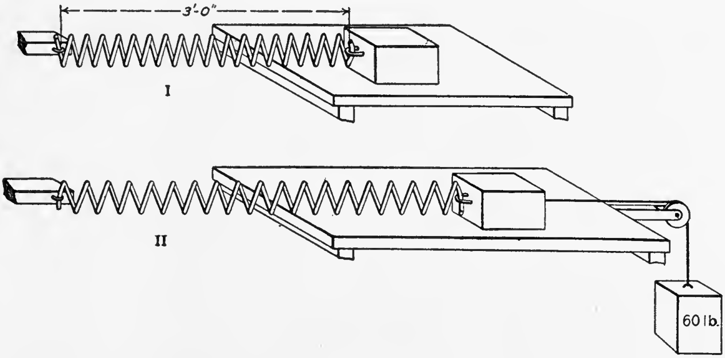

Fig. 250.

7. Figure 251 shows two springs attached to fixed points A and C, which are at the same level and are 10 feet apart. The natural length of each spring is 3 feet and a force of 20 pounds stretches each spring 1 foot. A body B is fastened to the free ends of these springs. What is the resultant force on the body when it is displaced 1 foot from the position of equilibrium, and when it is displaced x feet from that position? *Ans.* $K = -40x$.

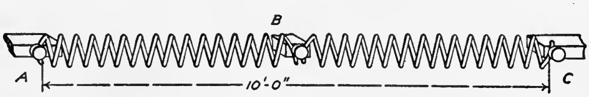

Fig. 251.

8. The two springs of Problem 7 are replaced by a single spring of natural length of 6 feet. It takes a force of 10 pounds to stretch this spring 1 foot. A body is attached 4 feet from the left end of the stretched spring. What force will displace this body 1 foot in the direction of the length of the spring?

183. Potential Energy of a Spring.—The work done by a force is the average force multiplied by the displacement. When the force varies as the displacement, the average force is the mean of the initial and final forces. If K is the force which produces unit deformation in a spring and s_1 is the initial deformation, the initial force is Ks_1. If s_2 is the final deformation, the final force is Ks_2. The average force for the displacement $s_2 - s_1$ is $\frac{K(s_1 + s_2)}{2}$.

$$U = \frac{K(s_2 + s_1)(s_2 - s_1)}{2} = \frac{K(s_2^2 - s_1^2)}{2}. \qquad (1)$$

Equation (1) gives the increase of potential energy.

Work is often represented by the area of a diagram in which force is the ordinate, and deformation or displacement is the abscissa. Figure 252 is the work diagram for a spring. The diagram is a trapezoid. The parallel sides are Ks_1 and Ks_2, and the distance between the sides is $s_2 - s_1$. If the initial displacement is zero, the diagram is a triangle and the total work is $\frac{Ks^2}{2}$.

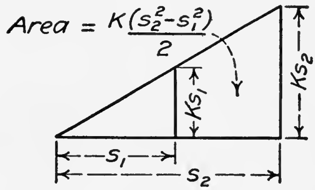

FIG. 252.

Problems

1. In Problem 1 of Art. 182, what is the work done in deflecting the board 6 inches? Solve without the formula.

2. In Problem 1 of Art. 182, what is the work done in deflecting the board 1 foot? What is the work of the last 6 inches of the foot? Construct the work diagram to the scale of 1 inch = 50 pounds and 1 inch = 0.25 ft.

184. Velocity Produced by Elastic Force.—When the force is proportional to the displacement, as in the case of a spring, the expression for the force is

$$P = -Kx, \qquad (1)$$

in which P is the force exerted by the spring, x is the displacement, and K is the force when the displacement is unity. The sign is negative for a spring because the direction of the force *exerted by the spring* is opposite the direction of the displacement. If a body of mass m is attached to the spring, as in Fig. 253, the acceleration of this body is given by Formula XXII.

$$-Kx = \frac{ma}{g}. \qquad (2)$$

Since acceleration along the X axis is $\frac{d^2x}{dt^2}$, Equation (2) may be written

$$-Kx = \frac{m}{g}\frac{d^2x}{dt^2}. \tag{3}$$

A differential equation of the form $\frac{d^2x}{dt^2}$ = *a function of* x is solved by first multiplying each side by dx.

$$-Kx\,dx = \frac{m\,dx}{g}\frac{d^2x}{dt^2} = \frac{m}{g}\frac{dx}{dt}\frac{d^2x}{dt}. \tag{4}$$

Since $\frac{dx}{dt} = v$, $\frac{d^2x}{dt} = dv$, and Equation (2) becomes

$$-Kxdx = \frac{m}{g}v\,dv. \tag{5}$$

Integrating Equation (5),

$$-\frac{Kx^2}{2} = \frac{mv^2}{2g} + C_1. \tag{6}$$

In Fig. 253, I, the mass m is at its position of equilibrium with equal tension on the two springs. In Fig. 253, II, the body

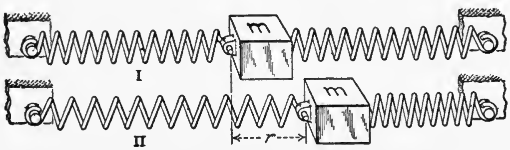

FIG. 253.

has been displaced a distance r toward the right and is assumed to be stationary. If let free at this point, the body will move to the position of Fig. 253, I, and will continue to move to a distance r on the left of the position of equilibrium. When $x = r$, $v = 0$. Substituting in Equation (6),

$$C_1 = -\frac{Kr^2}{2}.$$

A substitution for C_1 in Equation (6) gives

$$\frac{mv^2}{2g} = \frac{Kr^2}{2} - \frac{Kx^2}{2}. \qquad \text{Formula XXVI}$$

At the position of maximum deflection, at which $x = r$ and the velocity is zero, the body is at positive *elongation*. The distance r is called the *amplitude*.

The first member of Formula XXVI is the expression for kinetic energy, consequently the second member must also represent energy. The term $\frac{Kr^2}{2}$ gives the potential energy when the displacement is r. The term $\frac{Kx^2}{2}$ gives the potential energy when the displacement is x. The expression $\frac{Kr^2}{2} - \frac{Kx^2}{2}$ represents the work done on the spring when the displacements changed from x to r or the work done by the spring when the displacement is changed from r to x.

When the displacement is r feet, the mass m is stationary and all the energy is in the form of potential energy of the spring. When the displacement is reduced to x the potential energy of the spring is $\frac{Kx^2}{2}$. The difference between $\frac{Kr^2}{2}$ and $\frac{Kx^2}{2}$ represents the energy which is given up by the spring to the body and is transformed into kinetic energy of the mass. When $x = r$, all the energy is potential. When $x = 0$, all the energy is kinetic. When $x = -r$, all the energy is again potential and the body comes to rest with negative elongation equal to the positive elongation. This is the well known case of vibratory motion. The total motion is twice the amplitude. If there were no loss of energy, the vibration would continue indefinitely. Since there is always some loss of energy, the amplitude of the vibration will slowly diminish and the body will come to rest at the position of equilibrium after a long interval.

Formula XXVI might have been derived directly from the relation of the potential to the kinetic energy, without the use of integration. The integration method here given is general, however, and may be applied to any problem in which the effective force is a function of the displacement. It is advisable, therefore, to use the method in this case where it may be checked in order that it may be employed with confidence in other cases where it is not possible to verify the results.

Problems

1. A spring suspended from a fixed support is elongated 3 feet when a 24-pound mass is hung on it. The spring is stretched an additional 2 feet downward and then released. What will be the velocity when the spring is stretched 1 foot below the position of equilibrium? What will be the velocity when the body is 1 foot above the position of equilibrium? What will be the maximum velocity upward? What will be the maximum velocity downward? *Ans.* 5.67 ft. per sec.; 5.67 ft. per sec.; 6.54 ft. per sec.

2. A 24-pound mass rests on a horizontal table between the ends of two horizontal springs similar to those of Fig. 251. It requires a force of 20 pounds to move the body 1 foot in the direction of the length of the springs. The body is displaced a distance of 3 feet and then released. If there is no friction, find the maximum velocity of the body, the velocity when the displacement is 1 foot, and when the displacement is 2 feet.

Ans. 15.53 ft. per sec.; 14.65 ft. per sec.; 11.58 ft. per sec.

3. Solve Problem 2 if the coefficient of friction between the body and the table is 0.1.

When the displacement is 2 feet, the energy given up by the spring is $\frac{20(9-4)}{2} = 50$ ft.-lb. The work done in moving the body 1 foot against the friction is 2.4 ft.-lb. The kinetic energy is $50 - 2.4 = 47.6$ ft.-lb. The velocity of the 24-pound mass is 11.29 ft. per sec.

The maximum velocity is at the point at which the force of the spring is equal to the friction. At this point, the resultant force is zero and the acceleration is reduced to zero. This position is $\frac{2.4}{20} = 0.12$ feet from the point of zero tension in the spring.

$$\frac{24v^2}{2g} = \frac{20}{2}(9 - 0.0144) - 2.4 \times 2.88;$$

$$\max v = 14.91 \text{ ft. per sec.}$$

4. In Problem 3, find the maximum negative elongation from the position of zero force in the spring. *Ans.* $x = -2.76$ ft.

5. A mass of 40 pounds is placed on a small platform supported by a spring. The weight of the 40-pound mass elongates the spring a distance of 2 feet. The platform and mass are pulled down an additional 3 feet and then released. If no energy is lost and if the mass is not fastened to the platform, how high will the 40 pounds rise above the position of equilibrium?

Ans. 3.25 ft.

185. Vibration from Elastic Force.—From Formula XXVI, which applies to the force of a spring, or to any force which is proportional to the displacement and is opposite the direction of the displacement, the velocity is

$$v^2 = \frac{Kg}{m}(r^2 - x^2). \tag{1}$$

In this equation, v is the velocity when the displacement is x, r is the amplitude or maximum displacement, and K is the force in pounds exerted on the mass m when the displacement is 1 foot.

$$\frac{dx}{\sqrt{r^2 - x^2}} = \sqrt{\frac{Kg}{m}}\, dt; \tag{2}$$

$$\sin^{-1}\frac{x}{r} = \sqrt{\frac{Kg}{m}}\, t + C_2. \tag{3}$$

If the time of positive elongation at which $x = r$ is taken as the zero time,

$$C_2 = \sin^{-1} 1; \sin C_2 = 1; C_2 = \frac{\pi}{2}; \tag{4}$$

$$\sqrt{\frac{Kg}{m}}\, t = \sin^{-1}\frac{x}{r} - \frac{\pi}{2} = \cos^{-1}\frac{x}{r}; \tag{5}$$

$$\cos\left(\sqrt{\frac{Kg}{m}}\, t\right) = \frac{x}{r}; \tag{6}$$

$$x = r\cos\left(\sqrt{\frac{Kg}{m}}\, t\right). \tag{7}$$

Equation (6) gives the displacement of the mass in terms of the interval which has elapsed since it was at positive elongation.

When $x = r$, $t = 0$. When $x = 0$, $\cos\left(\sqrt{\frac{Kg}{m}}\, t\right) = 0$, and $\sqrt{\frac{Kg}{m}}\, t = \frac{\pi}{2}$. The time from positive elongation to zero elongation is one-half the time of single vibration and one-fourth the time of a complete period. From the expression above, the time from positive elongation to zero elongation is

$$t = \frac{\pi}{2}\sqrt{\frac{m}{Kg}}. \tag{8}$$

The time of a single vibration from positive elongation to negative elongation, or from zero elongation to positive elongation and back to zero elongation, is given by the equation

$$t = \pi\sqrt{\frac{m}{Kg}}. \tag{9}$$

The time of a complete period from positive elongation to positive elongation, or from any position to the same position with motion in the same direction, is given by the equation

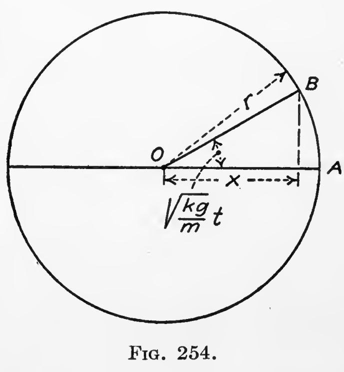

FIG. 254.

$$t_c = 2\pi\sqrt{\frac{m}{Kg}}. \tag{10}$$

Figure 254 represents a circle of radius r. The angle between the horizontal diameter and the radius OB is $\sqrt{\frac{Kg}{m}}\,t$ radians. The projection of OB on the horizontal diameter is

$$x = r \cos\left(\sqrt{\frac{Kg}{m}}\,t\right). \tag{11}$$

If the point B moves around the circle with uniform speed, $\sqrt{\frac{Kg}{m}}$ is the angle which the line OB passes through in one second, and if t is the time which has elapsed since the point B was at the right end of the horizontal diameter, then $\sqrt{\frac{Kg}{m}}\,t$ gives the angle which the line OB makes with the horizontal diameter at the end of time t. If t_c is the time of a complete revolution,

$$\sqrt{\frac{Kg}{m}}\,t_c = 2\pi;$$

$$t_c = 2\pi\sqrt{\frac{m}{Kg}}. \tag{12}$$

Equation (12) shows that the time of a complete revolution of the point B is the same as the time of a complete period of a vibrating mass.

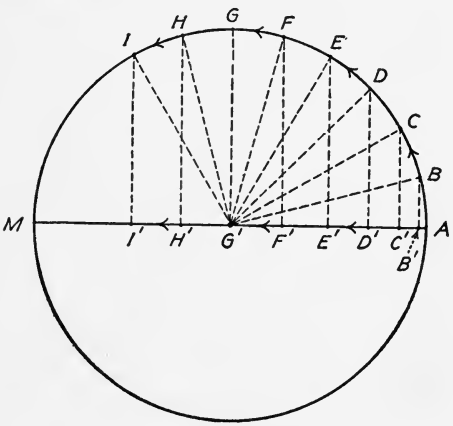

FIG. 255.

When a body is vibrating under the action of an attractive force which is proportional to the displacement, its motion along its path is the projection on that path of the motion of a body which moves with uniform speed around a circular path. In Fig. 255, A, B, C, D, E, etc. are equidistant points on the circumference of a circle. If a body moves in this circle with uniform speed, it passes from one of these points to the next one in the same interval of time. The points B', C', D', etc. are the projections of the points B, C, D, etc., on the horizontal diameter of the circle. The body which is vibrating along the straight line

AM passes from A to B', from B' to C', from C' to D', etc., in equal intervals of time.

A motion of this kind is called a *simple harmonic motion.* A simple harmonic motion is equivalent to the motion in the diameter of a circle of the projections of the positions of a body which moves with uniform speed in the circumference of that circle. The circle of Fig. 255 may be called the *reference circle* of the simple harmonic motion. The actual motion is along the straight line $A\ G'\ M$. The reference circle is used as a convenient device for finding the position and velocity at any given time.

If Equation (7) is differentiated with respect to time, the results are

$$\frac{dx}{dt} = -\sqrt{\frac{Kg}{m}}\, r \sin\left(\sqrt{\frac{Kg}{m}}\, t\right) = v_x; \tag{13}$$

$$\frac{d^2x}{dt^2} = -\frac{Kg}{m}\, r \cos\left(\sqrt{\frac{Kg}{m}}\, t\right) = -\frac{Kg}{m}\, x; \tag{14}$$

$$\frac{m}{g}\,\frac{d^2x}{dt^2} = \frac{ma}{g} = -Kx. \tag{15}$$

Equation (15) is identical with Equation (2) of Art. 183. The differentiation, therefore, checks the integration.

The velocity of the simple harmonic motion is a maximum when the sine of $\sqrt{\frac{Kg}{m}}\, t$ is equal to unity. From Equation (13), the maximum velocity is $r\sqrt{\frac{Kg}{m}}$. This is equal to the velocity of a body which is moving in the circumference of a circle of radius r and passing through the angle of $\sqrt{\frac{Kg}{m}}$ radians per second.

From Fig. 255, when the body which is moving in the reference circle $AG'M$ is passing the point G', its motion is parallel to the diameter AM. The projection of its velocity on the diameter is, therefore, equal to the velocity of the body which is moving in the circle of reference.

To find the position of a body which is moving with a simple harmonic motion, the maximum velocity is first calculated by means of the energy equation, Formula XXVI: A body is assumed to be moving with this velocity in the circumference of a reference circle. The radius of this reference circle is equal to the amplitude of the vibration. The time of one revolution of the body in the reference circle gives the time of a complete period of the vibration. The position at any time is found by the projection on the diameter of the reference circle.

Example

Find the time of a complete vibration of the body of Problem 2 of Art. 184. Find the position and velocity 0.2 second after the body is released.

The amplitude is 3 ft. and the circumference of the circle of reference is 18.8946 ft. The maximum velocity was found to be 15.53 ft. per sec. If t_c is the time of a complete period,

$$t_c = \frac{18.8496}{15.53} = 1.214 \text{ sec.}$$

At 0.2 second, the angle traversed in the reference circle is

$$\frac{0.2}{1.214} \times 360 = 59.31° = 59° \, 19'.$$

$$x = 3 \cos 59° \, 19' = 1.531 \text{ ft.}$$

$$v = 15.53 \sin 59° \, 19' = 13.35 \text{ ft. per sec.}$$

The time of a complete period may be checked by Equation (10) and the velocity by Equation (13). The velocity may also be checked by Formula XXVI after the displacement has been found.

It is recommended that the student make use of the reference circle and Formula XXVI instead of the equations of this article.

The potential energy of a stretched spring is proportional to the square of the displacement from the position of equilibrium. The kinetic energy is proportional to the square of the velocity. The maximum velocity is, therefore, directly proportional to the amplitude. The circumference of the reference circle is proportional to amplitude, consequently the time of vibration is independent of the amplitude. If a mass on a given spring has a maximum velocity of 1 foot per second when the amplitude of the vibration is 1 foot, it will have a maximum velocity of 2 feet per second when the amplitude of the vibration is 2 feet. In each case the time of a complete period is 2π seconds. Equation (12), which gives the time of a complete period, does not contain the amplitude. This fact is a mathematical proof of the statement that the time of vibration is independent of the length of the swing. A vibration which is independent of the amplitude is called an *isochronous* vibration.

Problems

1. The mass m of Fig. 253 is 40 pounds. It requires a pull of 16 pounds to displace the mass a distance of 1 foot in the direction of the length of the spring. The mass is displaced 2 feet from the position of equilibrium and then released. Find the maximum velocity. Find the time of a complete period. Find the velocity when the body is 1 foot from the position of equilibrium.

Ans. Max v = 7.175 ft. per sec.; t_c = 1.75 sec.

2. In Problem 1, find the maximum velocity and the time of a complete period if the mass is displaced 1 foot from the position of equilibrium and then released.

3. In Problem 1, the mass is displaced 2 feet and then released. Find the time required to come back 1 foot toward the position of equilibrium.

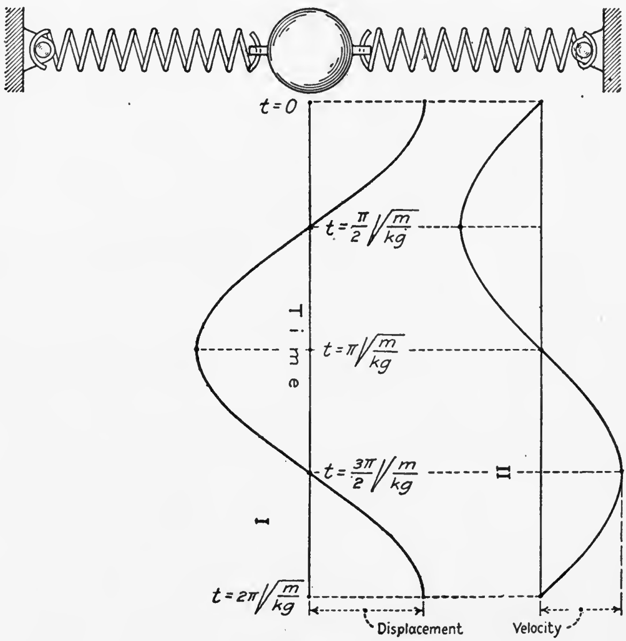

FIG. 256.

If a body vibrating with a simple harmonic motion is arranged to draw a line on a plate which moves with uniform speed at right angles to the direction of the vibration, the line so drawn will have the form of Fig. 256, I.

The equation for the velocity of a simple harmonic motion is

$$v = -\sqrt{\frac{Kg}{m}}\, r \sin\left(\sqrt{\frac{Kg}{m}}\, t\right). \tag{13}$$

The velocity curve is a sine curve which has the same period as the displacement curve of Fig. 256, I. The velocity is 0 when the displacement is a maximum. The velocity curve is shifted 90 degrees with reference to the displacement curve, as is shown in Fig. 256, II.

To an observer standing in front of an engine, the circular motion of the crank pin appears as a linear vertical motion, which is rapid at the middle and slower at the top and the bottom. This is a simple harmonic motion, if the engine is running at uniform speed. To an observer standing at one side of the engine, the piston rod and cross-head appear to move horizontally in a similar way. If the connecting rod were infinitely long,

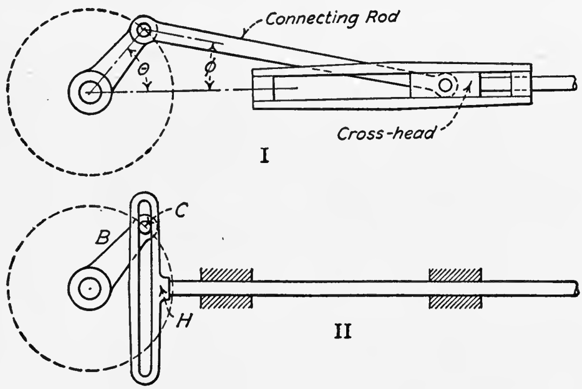

Fig. 257.

this would be a simple harmonic motion. With a connecting rod of finite length, as shown in Fig. 257, I, the motion of the cross-head differs slightly from a true simple harmonic motion.

Figure 257, II, shows a method of changing uniform circular motion into horizontal simple harmonic motion. The crank B carries a pin C which passes through a vertical slot in the head H. The horizontal component of the circular motion is transmitted to the horizontal rod. The motion of the rod is, therefore, a true simple harmonic motion, when the angular speed of the wheel is uniform.

186. Sudden Loads.—When a load is applied to an elastic body, the force exerted on the elastic body may be much greater than the weight of the load. Figure 258 represents a helical spring on which is placed a mass of m pounds. Figure 258, I, shows the natural length of the spring. In Fig. 258, II, the mass

m is attached to the spring but is held up by the support B so that none of its weight comes on the spring. If the support B is lowered slowly until all the weight comes on the spring, the condition of Fig. 258, III, is reached. The mass comes to rest

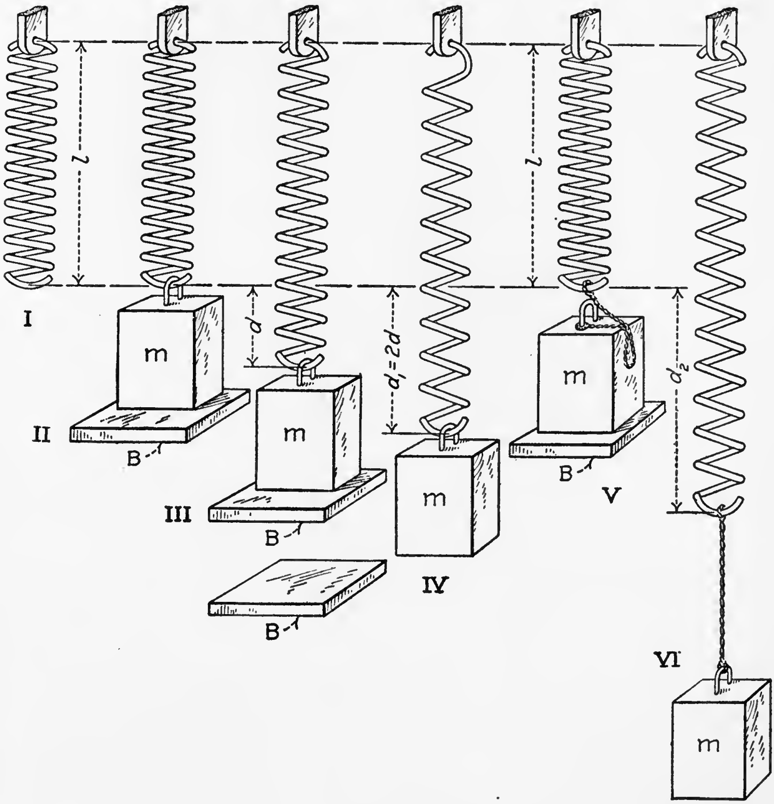

FIG. 258.

when the spring is stretched a distance d. If K pounds is sufficient to stretch the spring 1 foot,

$$d = \frac{m}{K}. \tag{1}$$

The work done by gravity on the mass m while it moves a distance of d feet is

$$U = m\,d = \frac{m^2}{K}. \tag{2}$$

The work done by the mass on the spring, or the potential energy acquired by the spring, is

$$U = \frac{0 + m}{2} d = \frac{md}{2} = \frac{m^2}{2K}. \tag{3}$$

Equation (3) represents only half the work of Equation (2). It is necessary to account for the remainder. At the beginning of the descent, the entire weight of the body rested on the support B. As the body was lowered, the spring took an increasingly greater proportion until it finally carried the entire load and the reaction of the support became zero. The work done on the support B was

$$\frac{m + 0}{2} d = \frac{md}{2}. \tag{4}$$

Equation (4) accounts for the remaining half of the energy.

If the support B is suddenly removed when there is no tension in the spring, the motion is that shown in Fig. 258, IV. No work is done on the support and all the energy is stored in the spring. If d_1 is the total elongation of the spring, the average tension is $\frac{Kd_1}{2}$ and the potential energy of the spring is $\frac{Kd_1^2}{2}$. The work of gravity on the mass is md_1. Equating the work and energy,

$$md_1 = \frac{Kd_1^2}{2}; \tag{5}$$

$$d_1 = \frac{2m}{K}. \tag{6}$$

A comparison with Equation (1) shows that the elongation when the load is "suddenly applied" is twice as great as when the load is gradually applied. When a load is suddenly applied to a spring or other elastic body in which the stress is proportional to the deformation, the deformation is twice as great as that due to the weight of the load. Since the force is proportional to the deformation, it follows that the force is likewise twice as great as the weight of the load. This fact is expressed by the statement "Live load is twice the dead load."

Since the tension in the spring at the bottom of its path in Fig. 258 is twice the weight of the body, the resultant force *upward* in that position is m pounds. The body will move upward to the position of Fig. 258, II, and will continue to vibrate for a long time. It will finally come to rest in the position of Fig. 258, III.

In Fig. 258, V, the body is suspended from the spring by an inelastic cord which permits it to fall a distance h before it brings any tension on the spring. If d_2 is the distance which the spring is stretched, the work of gravity on the mass is $m(h + d_2)$, and the work done on the spring is $\frac{Kd_2^2}{2}$.

$$m(h + d_2) = \frac{Kd_2^2}{2}. \tag{7}$$

When a load falls upon an elastic body, the deformation is greater than twice the deformation caused by the weight of the load. The body will vibrate according to the laws of a simple harmonic motion from the position of no deformation of the elastic body down to the lowest point and back again. After it reaches the position at which there is no deformation of the elastic body, the load will continue to move as a projectile which is thrown straight upward. After reaching the starting point, it will fall and repeat the cycle.

Problems

1. A force of 10 pounds stretches a given spring 6 inches. How much is this spring stretched when a mass of 40 pounds is hung on it so that the load is gradually applied?

2. How much is the spring of Problem 1 stretched if the mass of 40 pounds is put on suddenly? What is the tension at the lowest point? What is the time required to reach the lowest point?

3. The 40-pound mass of Problem 1 falls on the spring from a height of 1 foot. How much is the spring elongated? *Ans.* 4.828 ft.

4. In Problem 3, what is the entire time required for the mass to fall, compress the spring, and return to its original position?

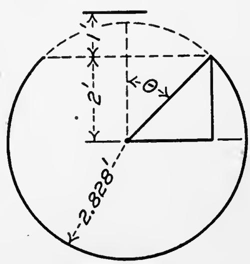

FIG. 259.

At the position of equilibrium, the spring is stretched 2 feet. The amplitude of the vibration below this point is 2.828 feet. Consequently, the radius of the circle of reference is 2.828 feet. The angle θ of Fig. 259 is given by the equation

$$\cos \theta = \frac{2}{2.828}$$

$$\theta = 45°.$$

If the entire vibration were a simple harmonic motion, the time of the complete period would be $t_c = 1.566$ seconds. The time required to traverse

270 degrees of the reference circle is 1.174 seconds. The time required to fall 1 foot is 0.249 second. The entire time is 1.174 + 0.498 = 1.672 sec.

187. Composition of Simple Harmonic Motions.—Figure 260 shows a body supported by a single vertical spring and held by two horizontal springs. [One horizontal spring is sufficient if it does not buckle in compression.] The springs are supposed to be so long that the vertical component of the tension in the horizontal springs when the body is displaced vertically may be neglected, and the horizontal component in the vertical spring when the body is displaced horizontally may be also neglected. If the body is displaced horizontally, it will vibrate in that direction with a simple harmonic motion. If it is displaced vertically,

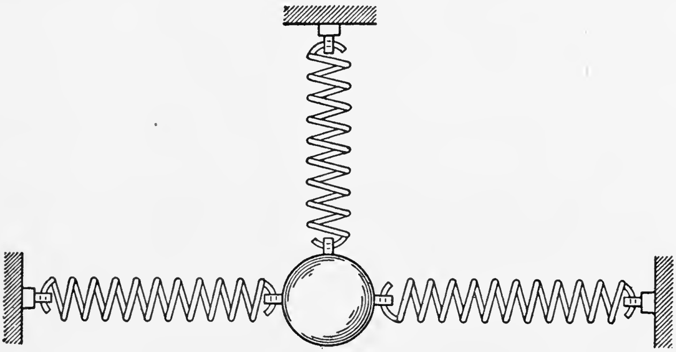

FIG. 260.

it will vibrate vertically with a simple harmonic motion. If it is displaced to the right a distance a and given a vertical component upward sufficient to carry it to a vertical distance b, it will then have a motion which is the resultant of the two simple harmonic motions at right angles to each other. If K_h is the constant of the horizontal springs, and K_v is the constant of the vertical spring, the equations of displacement are

$$x = a \cos \sqrt{\frac{K_h g}{m}}\, t, \tag{1}$$

$$y = b \sin \sqrt{\frac{K_v g}{m}}\, t. \tag{2}$$

If $K_h = K_v$, the periods of the two components are equal and the path of the body is an ellipse. If the amplitudes are equal, the ellipse becomes a circle.

Figure 261 shows the motion when the time of vibration in the horizontal direction is twice as great as in the vertical direction. The equations may be written briefly,

$$x = a \sin ct;\ y = b \sin 2ct.$$

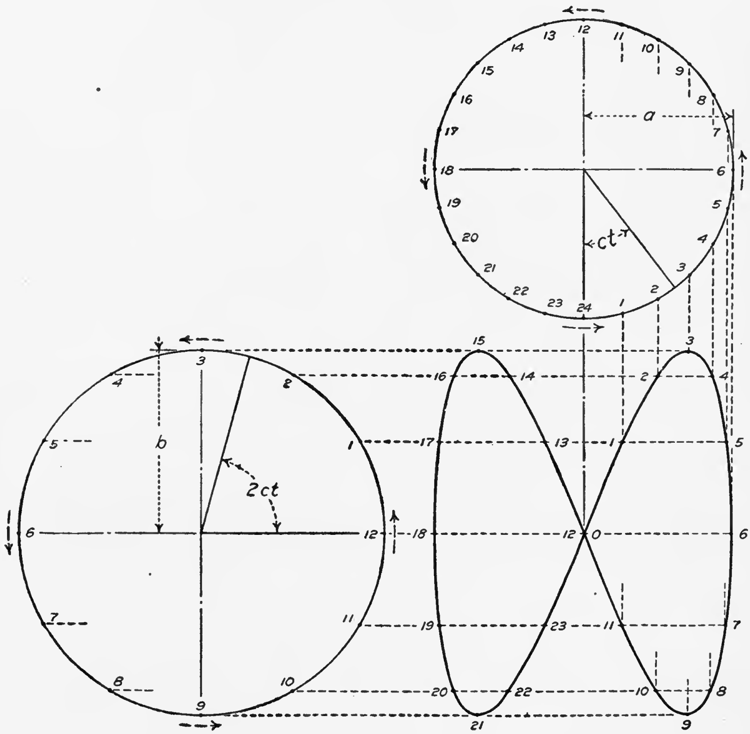

FIG. 261.

Problems

1. If the force required to stretch the horizontal spring 1 foot is 20 pounds, what must be the force required to stretch the vertical spring 1 foot in order that the time of vibration vertically may be one-half the time of vibration horizontally?

2. Plot the curve of the resultant of two simple harmonic motions of equal amplitude if the time of vibration horizontally is two-thirds the time of vibration vertically.

Instead of the springs as shown in Fig. 260, a single bar may be used as the elastic body. The bar is fixed rigidly at one end and carries a heavy mass at the other end. The section of the bar is a rectangle. In Fig. 262, the vertical sides of the rectangle are greater than the horizontal width. The vertical stiffness is greater than the horizontal stiffness and the time of vibration in the vertical plane is greater than in the horizontal plane. These dimensions may be such as to give any desired ratio to the time of vibration.

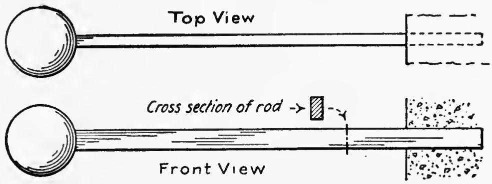

FIG. 262.

Problem

(A knowledge of Strength of Materials is required for this problem.)

3. A steel cantilever 30 inches long is ¼ inch wide and ⅜ inch deep. It carries a mass of 2 pounds on the free end. If $E = 30{,}000{,}000$, find the force required to deflect this cantilever 1 inch in a vertical direction and the force required to deflect it 1 inch in the horizontal direction. Calculate K_v and K_h. Neglecting the mass of the bar, find the time of a complete period in each direction.

188. Correction for the Mass of the Spring.—In the case of a vibrating spring or other elastic body, a part of the kinetic energy is located in the spring. For accurate work, correction must be made for this energy.

The effective mass of the spring may be calculated from two sets of vibration experiments. Let m_0 be the effective mass of the spring, m_1 be a known additional mass, and m_2 be a second known mass. The time of vibration is found with the mass m_1 on the spring, and again with m_2 on the spring. If t_1 and t_2 are these times of vibration,

$$t_1 = 2\pi\sqrt{\frac{m_0 + m_1}{Kg}}; \tag{1}$$

$$t_2 = 2\pi\sqrt{\frac{m_0 + m_2}{Kg}}; \tag{2}$$

$$\frac{t_1^2}{t_2^2} = \frac{m_0 + m_1}{m_0 + m_2}; \tag{3}$$

$$\frac{m_0 + m_1}{m_2 - m_1} = \frac{t_1^2}{t_2^2 - t_1^2}. \tag{4}$$

Problem

The time of vibration of a given spring with a load of 2 pounds is 1.640 seconds. The time of vibration with a load of 10 pounds is 2.880 seconds. Find the effective mass of the spring.

Ans. $m_0 = 1.839$ lb.

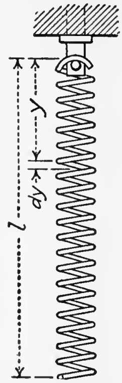

FIG. 263.

The effective mass of a helical spring may easily be calculated. Let m be the mass of a helical spring of length l, as shown in Fig. 263. The mass per unit length is $\frac{m}{l}$. The mass of an element of length dy is $\frac{mdy}{l}$. If v is the velocity of the free end of the spring at a distance l from the fixed end, the velocity of an element at a distance y from the fixed end is $\frac{vy}{l}$. The energy of the element is given by the equation

$$dU = \frac{mv^2}{2gl^3} y^2 dy; \tag{5}$$

$$U = \frac{mv^2}{2gl^3} \left[\frac{y^3}{3} \right]_0^l; \tag{6}$$

$$U = \frac{mv^2}{6g}. \tag{7}$$

Equation (7) shows that the effective mass of a *uniform* helical spring is one-third the mass of the spring. This statement applies to the helical part of the spring. Where the spring bends toward the axis at the ends, the ratio is different. In a long spring with a large number of turns, the error which is due to the end connections may be neglected.

189. Determination of g.—The vibration of a body on a helical spring may be used for the determination of g. The time of vibration is determined with a known mass on the spring; the elongation of the spring under a known load is measured; and the spring is weighed. The results of these measurements afford the necessary data for the calculation of gravity.

EXAMPLE

A spring which weighs 0.24 pound is elongated 0.421 ft. when a load of 5 pounds is applied. With the 5-pound mass on the spring, the time of

100 complete periods is 72.4 seconds. Find the value of g for that locality.

$$t_c = 2\pi\sqrt{\frac{m}{Kg}};$$

$$g = \frac{4\pi^2 m}{Kt_c^2}$$

$$K = \frac{5}{0.421}; \quad m = 5 + 0.08; \quad t_c = 0.724 \text{ sec.}$$

$$g = \frac{4\pi^2 \times 5.08 \times 0.421}{5 \times 0.724^2} = 32.20 \text{ ft.}$$

This experiment does not require that the masses be accurately known in terms of the standard units. If the effective mass of the spring were negligible, it would only be necessary to find the elongation due to any given mass and the time of vibration with that mass. Since the effective mass of the spring is not negligible, the ratio of its mass to that of the additional load must be determined. It is not necessary, however, to determine the actual mass of either the spring or the load.

Problem

A helical spring weighing 0.42 pound is stretched 0.362 foot by a load of 6 pounds. With a mass of 10 pounds on the spring, the time of 100 complete periods is 86.6 seconds. Find g.

190. Determination of an Absolute Unit of Force.—If an absolute unit of force is used, Equations (2) and (3) of Art. 184 become

$$-K'x = ma; \tag{1}$$

$$-K'x = m\frac{d^2x}{dt^2}. \tag{2}$$

In these equations, K' is the force in absolute units which causes unit displacement.

Formula XXVI is

$$\frac{mv^2}{2} = \frac{K'r^2}{2} - \frac{K'x^2}{2}. \tag{3}$$

Equation (12) of Art. 185 is

$$t_c = 2\pi\sqrt{\frac{m}{K'}}. \tag{4}$$

Equation (4) affords an easy method of finding the constant of the spring and getting a unit of force which is entirely independent of gravity.

Problems

1. A helical spring weighs 0.6312 pound. A 10-pound "weight" is hung on this spring and the time of 1000 complete periods is found to be 1243.6 seconds. Find the constant of the spring in poundals if the actual mass of the 10-pound weight is 9.9842 pounds.

Ans. $K' = 260.2$ poundals; 100 poundals elongates the spring 0.3844 ft.

2. A given spring makes 1000 complete vibrations in 528.3 seconds when a 500 gram mass is attached to it, and makes 1000 complete vibrations in 1014.2 seconds when the mass is increased to 2000 grams. How many dynes force are required to deform this spring 1 centimeter?

Ans. $K' = 7900.9$ dynes.

191. Positive Force which Varies as the Displacement.—The reaction of a spring is proportional to the displacement. The direction of the reaction is opposite the direction of the displacement. The expression for the reaction of the spring is $P = -Kx$. When the force varies as the displacement and is in the same direction, the expression is

$$P = Kx. \tag{1}$$

If a body subjected to a force of this kind is in equilibrium, the equilibrium is unstable. If the body is displaced ever so little from its position of equilibrium, the increased force in the direction of the displacement will cause it to continue to move in the same direction, and it will not return to the position of equilibrium. If the body is displaced a distance r from the position of equilibrium, the work done by the force is $\frac{Kr^2}{2}$. The work done in displacing the body from a distance r to a distance x from the position of equilibrium is

$$U = \frac{K}{2}(x^2 - r^2). \tag{2}$$

If the body is at rest when the displacement is r, its kinetic energy when its displacement is x is given by the equation

$$\frac{mv^2}{2g} = \frac{K}{2}(x^2 - r^2). \tag{3}$$

Problems

1. A rope, 100 feet long, weighing 1 pound per foot, runs over a smooth pulley as shown in Fig. 264. When the center of the rope is 1 foot below the pulley, so that there is 49 feet of rope on one side of the pulley and 51 feet on the other side, what is the resultant force? *Ans:* $P = 2$ lb.

2. In Problem 1, what is the expression for the resultant force in terms of the displacement of the center of the rope? *Ans.* $P = 2x$.

3. Find the work done when the rope of Fig. 264 moves from the position of equilibrium to the position at which there is 40 feet of rope on one side and 60 feet on the other. *Ans.* $U = 100$ ft.-lb.

4. What is the final velocity in Problem 3 if the initial velocity is so small that its kinetic energy is negligible? *Ans.* $v = 8.02$ ft. per sec.

FIG. 264.

The relation of displacement to time is found by integration. From Equation (3)

$$\frac{dx}{dt} = v = \sqrt{\frac{Kg}{m}}\sqrt{x^2 - r^2}; \tag{4}$$

$$\frac{dx}{\sqrt{x^2 - r^2}} = \sqrt{\frac{Kg}{m}}\,dt; \tag{5}$$

$$\log\left(x + \sqrt{x^2 - r^2}\right) = \sqrt{\frac{Kg}{m}}\,t + C. \tag{6}$$

If time is measured from the time when $x = r$, then $t = 0$ when $x = r$. Substituting in Equation (6),

$$C = \log r; \tag{7}$$

$$\log\frac{x + \sqrt{x^2 - r^2}}{r} = \sqrt{\frac{Kg}{m}}\,t; \tag{8}$$

$$t = \sqrt{\frac{m}{Kg}}\log\frac{x + \sqrt{x^2 - r^2}}{r}. \tag{9}$$

Problem

5. In Fig. 264, the rope is placed with 49 feet on one side and 51 feet on the other and then released. Find the time required for the rope to entirely leave the pulley. Find the final velocity.

Ans. $t = 5.63$ sec.; $v = 40.10$ ft. per sec.

192. Summary.—There are two classes of forces which vary as the displacement. In one class, the direction of the force is opposite the direction of the displacement. In the other class, the direction of the force is the same as the direction of the displacement.

A spring or other elastic body exerts a force which is opposite the displacement. The expression for the force is

$$P = -Kx.$$

The change of kinetic energy when the body moves from displacement r to displacement x is $\frac{Kr^2}{2} - \frac{Kx^2}{2}$. If the velocity is zero at displacement r, the velocity at displacement x is given by the energy equation,

$$\frac{mv^2}{2g} = \frac{Kr^2}{2} - \frac{Kx^2}{2}. \qquad \text{Formula XXVI}$$

When the force is proportional to the displacement and in the opposite direction, the motion is periodic. The position at any time is given by the equation

$$x = r \cos \sqrt{\frac{Kg}{m}}\, t,$$

in which r is the maximum displacement, or the amplitude of the vibration, t is the time which has elapsed since the displacement was equal to r, and x is the displacement at time t. The velocity at time t is given by the equation

$$v = \frac{dx}{dt} = -\sqrt{\frac{Kg}{m}} \sin \sqrt{\frac{Kg}{m}}\, t.$$

The time of a complete period is

$$t_c = 2\pi \sqrt{\frac{m}{Kg}}$$

The time of vibration is independent of the amplitude.

Simple harmonic motion may be studied by means of a reference circle, the radius of which is equal to the amplitude of the motion. The maximum velocity is found by Formula XXVI. A body is assumed to move with this velocity in the circumference of the reference circle. The projection of the motion on the diameter of the reference circle gives the required motion.

When a load is applied suddenly to an elastic body, the deformation is twice as great as that when the load is applied gradually, and the stress is twice as great as that resulting from a quiescent load.

An absolute unit of force may be determined by means of the time of vibration of a known mass on a spring. The value of g may be found by means of the time of vibration of a mass on a spring and the elongation of the spring caused by the weight of the mass.

The second class of force which is proportional to the displacement is expressed by the equation

$$P = Kx.$$

If the velocity is zero at displacement r, the velocity at displacement x is given by the energy equation

$$\frac{mv^2}{2g} = \frac{Kx^2}{2} - \frac{Kr^2}{2}.$$

The motion is not periodic. The equilibrium when $x = 0$ is unstable. Equations (8) and (9) of Art. 191 give the position in terms of the time.

CHAPTER XX

CENTRAL FORCES

193. Definition.—A central force is one whose direction is along a line which joins its point of application wtih some fixed point; whose magnitude depends upon the length of that line and does not depend upon its direction. In Fig. 265, O is a central point

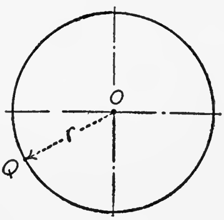

Fig. 265.

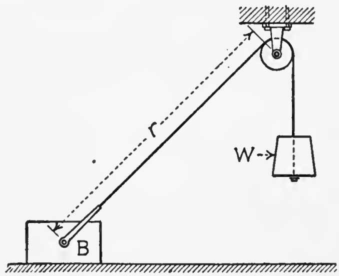

Fig. 266.

and Q is the point of application of a force directed toward O. If the force satisfies the definition of a central force, its magnitude is constant for any position of Q on the surface of a sphere of radius OQ with its center at O.

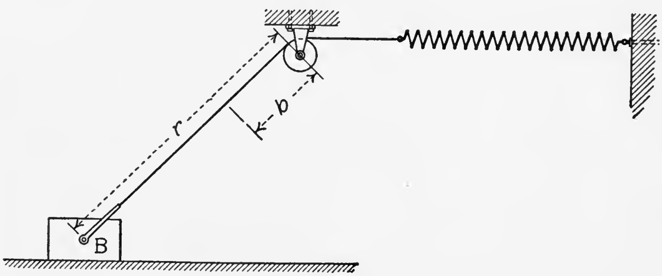

Fig. 267.

In Fig. 266, a rope which supports a load W runs over a small pulley. The force at B is constant and is directed toward the pulley as a center.

In Fig. 267, the rope which runs over the pulley is attached to a spring. If the end of the rope is at a distance b from the pulley when there is no tension in the spring, the force at B is given by the equation,

$$P = -K(r - b). \tag{1}$$

The force of gravity is a central force directed toward the center of the Earth. Above the surface of the Earth, the magnitude of the force of gravity varies inversely as the square of the distance from the center. The force is given by the expression,

$$P = -\frac{K}{r^2}. \tag{2}$$

An expression similar to Equation (2) gives the attraction or repulsion of a single magnetic pole or of an electrically charged sphere.

194. Work of a Central Force.—The work done by a central force depends upon the distances of the initial and final points of application from the center, and is independent of the form of the path.

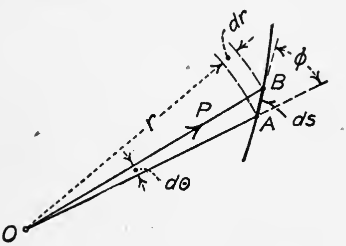

FIG. 268.

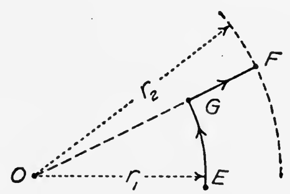

FIG. 269.

In Fig. 268, the point of application moves a distance ds from point A to point B. The angle between the radius vector OA and the line AB is ϕ. The work of the displacement is

$$dU = P \cos \phi \, ds = Pds \cos \phi. \tag{1}$$

Since $ds \cos \phi = dr$,

$$dU = Pdr; \tag{2}$$

$$U = \int Pdr. \tag{3}$$

If a body under the action of a force which is central at O, Fig. 269, moves from a point E at a distance r_1 from O to a point F at a distance r_2 from O, the total work depends only upon the distance $r_2 - r_1$. The body might move from E to G on the circumference of a circle of radius r_1. Since the force is normal to the displacement, no work is done during this displacement. The body may then move along the line *OGF* directly away from the center. The total work is equal to that given by Equation (3).

Problems

1. A body rests on a horizontal plane. It is subjected to a constant force of 20 pounds directed toward a point *O* which is 12 feet above the plane, and directly over a point *B* on the plane. The body moves from a point which is 16 feet from *B* to a point which is 5 feet from *B*. Find the work done by the force of 20 pounds. Solve by arithmetic.

2. Solve Problem 1 by integration, assuming that the body moves along a straight line on the surface of the plane.

From Fig. 270,

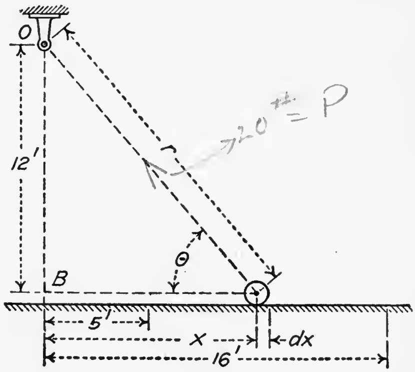

FIG. 270.

$$dU = 20 \cos \theta \, dx = -\frac{20 \times dx}{\sqrt{144 + x^2}};$$

$$U = 140 \text{ ft.-lb.}$$

3. The body of Problem 1 weighs 60 pounds. If its velocity is zero when it is 16 feet from *B*, what is its velocity when it is 5 feet from *B*? If its velocity is 20 feet per second when it is 16 feet from *B*, what is its velocity when it is 5 feet from *B*?

Ans. $v = 12.26$ ft. per sec.; $v = 23.45$ ft. per sec.

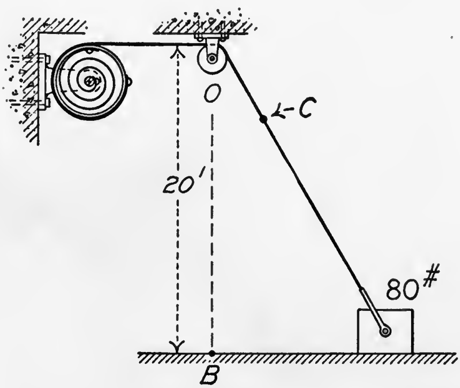

FIG. 271.

4. The body of Problem 1 weighs 60 pounds and the coefficient of friction between the body and the plane is 0.1. Find the increase of kinetic energy when the body moves from the point at 16 feet from *B* to the point at 5 feet from *B*. Find the friction by integration.

Ans. 94.63 ft.-lb.

5. Figure 271 shows a drum which is rotated by a spiral spring. A rope attached to the drum runs over a smooth pulley *O* at a distance of 20 feet above a point *B* on a smooth horizontal floor. There is no tension in the rope when the end is at *C* at a distance of 5 feet from *O*. A force of 3 pounds is required to rotate the drum 1 foot. The drum is rotated and the end of the rope is fastened to a mass of 80 pounds which rests on the floor. The 80-pound mass moves from a point which is 21 feet from B to a point which is 15 feet from B. Find the work done by the spring. If the mass of the drum and spring is negligible and the 80-pound mass starts from rest, find its final velocity. *Ans.* $U = 264$ ft.-lb.

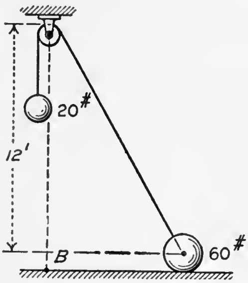

FIG 272.

6. Solve Problem 5 if the coefficient of friction between the mass and the floor is 0.12.

Ans. Change of kinetic energy = 2 64 − 57.6 + 43.2 − 8.03 = 241.57 ft.-lb.

7. Figure 272 is similar to Fig. 270. The force, however, is that resulting from a suspended mass of 20 pounds. The body starts from a point 16 feet from B. If there is no friction, find its velocity when it reaches B.

Ans. $v = 13.1$ ft. per sec.

8. In Fig. 272, find the velocity of the 60-pound mass at 5 feet from B if its velocity was zero at 16 feet from B. Find also the velocity of the 20-pound mass.

Ans. Velocity of the 60-pound mass = 11.91 ft. per sec.

195. Force Inversely as the Square of the Distance.—When the magnitude of the force varies inversely as the square of the distance from the central point and is directed toward that point, its equation is

$$P = -\frac{K}{r^2}. \tag{1}$$

When the point of application is displaced a distance dr, the increment of work done by the force is

$$dU = -\frac{Kdr}{r^2}. \tag{2}$$

If the total displacement is $r_2 - r_1$, the total work is

$$U = \left[\frac{K}{r}\right]_{r_1}^{r_2} = -K\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \tag{3}$$

Problems

1. The attraction between two charged spheres is 200 dynes when their centers are 5 centimeters apart. Find K of the equations above. How much work is done in moving one sphere from a position 10 centimeters from the other to a position 20 centimeters from the other?

Ans. $K = 5000$ dynes; $U = 500$ ergs.

2. For bodies above the Earth's surface, the attraction of gravitation varies inversely as the square of the distance from the center of the Earth. If the radius of the Earth is taken as 4000 miles, and is expressed in feet, what is the value of K in pounds? *Ans.* $K = 21{,}120{,}000^2$.

3. A mass of 1 pound is lifted from the surface of the Earth a vertical distance of 4000 miles. Find the work in foot-pounds.

Ans. $U = 10{,}560{,}000$ ft.-lb.

4. If only the attraction of the Earth is considered, show that the work done in moving a body from the surface of the Earth to an infinite distance is 21,120,000 foot-pounds multiplied by the mass of the body.

5. A body falls to the Earth from an infinite distance. If the resistance of the air were negligible, with what velocity would it strike the Earth?

Ans. 37,000 ft. per sec.

If a body were thrown out of the Earth's atmosphere with a velocity of 37,000 feet per second, it would not return. It would not, however, leave the solar system on account of the greater energy required to overcome the attraction of the Sun.

The results of Problems 2, 3, and 4 are slightly in error owing to the fact that 4000 miles has been used as the radius of the Earth. The actual radius is less than 21,000,000 feet. (See Hussey's Mathematical Tables, p. 147.)

196. Equipotential Surfaces.—In Art. 135, an equipotential surface was defined as a surface normal to the resultant of all the applied forces except the reaction of the surface. No work is done when a body moves from one part of an equipotential surface to another. When the forces have a single center, the equipotential surfaces are concentric spheres.

If U is the potential energy of a body, and if the body receives a displacement dx while y and z remain constant, the change in the energy is the partial derivative of U with respect to x multiplied by dx. When x alone varies,

$$\text{increment of } U = \frac{\partial U}{\partial x}\,dx. \tag{1}$$

Since the work done in moving a distance dx is the product of dx multiplied by the component of the force in that direction, the partial derivative of U with respect to X is the X component of the applied forces.

$$\frac{\partial U}{\partial x} = H_x;\ \frac{\partial U}{\partial y} = V;\ \frac{\partial U}{\partial z} = H_z. \tag{2}$$

The equation of the equipotential surface is

$$\frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy + \frac{\partial U}{\partial z}\,dz = 0. \tag{3}$$

The potential energy U is called the *potential function.* The potential function may be defined as the expression whose derivative with respect to any direction gives the force in that direction.

Problems

1. Solve Problem 1 of Art. 133 by Equation 3. Assume that the diameter of the pulley is negligible.

If the origin of coördinates is taken at the pulley, Fig. 273, the energy due to gravity is $10y$. The energy of the spring is $\frac{r^2}{2} = \frac{x^2 + y^2}{2}$

$$U = 10y + \frac{x^2 + y^2}{2};$$
$$\frac{\partial U}{\partial x} = x; \frac{\partial U}{\partial y} = 10 + y;$$
$$xdx + (10 + y)dy = 0;$$
$$\frac{x^2}{2} + \frac{(10 + y)^2}{2} = C.$$

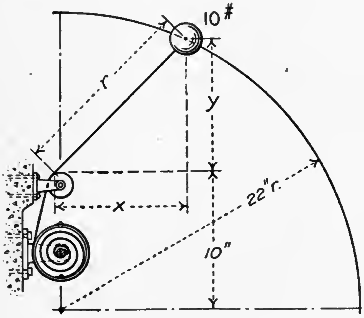

FIG. 273.

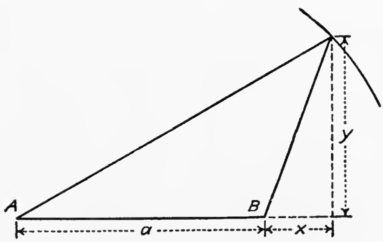

FIG. 274.

2. A body is subjected to the action of two forces directed toward two fixed points, A and B of Fig. 274, at a distance a apart. The magnitude of each force is Kr in which r is the distance from A for one force and the distance from B for the other. Find the equation of the equipotential curves in a plane through A and B.

If B is taken as the origin of coördinates,

$$2U = K((a + x)^2 + y^2) + K(x^2 + y^2);$$
$$[2(a + x) + 2x]dx + [2y + 2y]dy = 0;$$
$$adx + 2xdx + 2ydy = 0;$$
$$ax + x^2 + y^2 = C.$$

The curves are circles with their centers midway between A and B.

3. In Fig. 274, let the distance from A to B be 10 feet. Let the force which is central at A be $3r$, where r is the distance from A; and let the force which is central at B be $2r$, where r is the distance from B. With the origin of coordinates at B and with the X axis passing through A, find the equation of the equipotential curve through the point (0,8).

Ans. $(x + 6)^2 + y^2 = 100.$

The curve is a circle with its center on the line AB at a distance of 6 feet from B.

4. In Problem 3, find the potential energy at the point (0,8) and at the point on the X axis at the same distance from the center of the circle Compare the two values.

197. Summary.—The direction of a central force is along the line which joins its points of application with some center; its magnitude is some function of the length of this line.

The work done by a central force depends only upon the difference of the distances of the initial and final positions from the center.

$$U = P\int dr$$

When the magnitude of a force varies inversely as the square of distance of its point of application from the center, the potential difference between two points is

$$U = K\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

If r_2 is infinity,

$$U = \frac{K}{r},$$

which is the potential of the point. If the force is attraction, the potential is the work done by the force in bringing the body from infinity to the point at a distance r from the center. If the force is repulsion, the potential is the work done against the force in bringing the body from infinity.

The potential due to several forces is found by adding the potential due to the separate forces. Since potential is not a vector, this addition is the addition of ordinary arithmetic. The derivative of the potential with respect to any direction gives the force in that direction,

$$\frac{\partial U}{\partial x} = H_x;\ \frac{\partial U}{\partial y} = V;\ \frac{\partial U}{\partial z} = H_z.$$

The equation of an equipotential surface is

$$\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz = 0$$

CHAPTER XXI

ANGULAR DISPLACEMENT AND VELOCITY

198. Angular Displacement.—When a rigid body rotates about an axis, all parts of the body turn through the same angle. In Fig. 275, the axis passes through O and is perpendicular to the plane of the paper. If a point on the body moves from position B to position B', its path is the arc BB'. The angular displacement of this point is the angle between the radii OB and OB'. If r is the length of the radius from O to B, the angular displacement in radians is

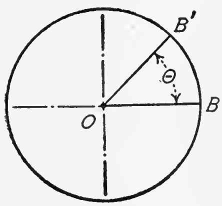

FIG. 275.

$$\theta = \frac{\text{arc } BB'}{r}.$$

Angular displacement is linear displacement divided by the radius.

Problems

1. A point moves a distance of 6 feet in the circumference of a circle of 3-foot radius. Find the angular displacement in radians.

Ans. $\theta = 2$ radians.

2. A body revolves once on its axis. What is the angular displacement?

Ans. $\theta = 2\pi = 6.2832$ radians.

3. What is the angular displacement when a body turns through an angle of 60 degrees?

4. A cylinder 4 feet in diameter turns through an angle of 50 degrees. Find the linear displacement of any point on the surface of this cylinder.

5. The hour hand of a clock moves from 2 to 10. What is the angular displacement?

199. Angular Velocity.—Angular velocity is the rate of change of angular displacement. Angular velocity is usually represented by the Greek letter ω (omega).

$$\omega = \frac{\theta_2 - \theta_1}{t_2 - t_1}. \tag{1}$$

Equation (1) is analogous to the corresponding formula for linear velocity. For small intervals of time,

$$\omega = \frac{d\theta}{dt}. \tag{2}$$

If v is the linear velocity of a point on a rotating body, vt is the length of the arc which the point describes in time t. The angular displacement of the point is $\frac{vt}{r}$. The angular velocity of the point is

$$\frac{vt}{r} \div t = \frac{v}{r} = \omega; \tag{3}$$

$$v = r\omega \qquad \text{Formula XXVII.} \tag{4}$$

The angular velocity of a point is its linear velocity divided by the radius; the linear velocity is the angular velocity multiplied by the radius.

Problems

1. The circumference of a flywheel 6 feet in diameter is moving 3000 feet per minute. What is the angular velocity of the wheel?

Ans. $\omega = 16.67$ radians per sec.

2. What is the angular velocity of a pulley which is making 300 revolutions per minute? *Ans.* $\omega = 10\pi = 31.416$ radians per sec.

3. A steel ball is whirled in a vertical plane on the end of a rod 4 feet in length at the rate of 8 revolutions in 4 seconds. If the ball is released from the rod when the rod is horizontal and is moving upward, how high will the ball rise? How many times will the ball rotate on its axis before it returns to the starting point?

The formula for displacement in simple harmonic motion is

$$x = r \cos \sqrt{\frac{Kg}{m}}\, t.$$

The expression $\sqrt{\frac{Kg}{m}}t$ is the angular displacement in the circle of reference, and $\sqrt{\frac{Kg}{m}}$ is the angular velocity in that circle. The time of a complete period, which corresponds with the time of one revolution in the circle of reference, is

$$t_c = 2\pi \div \sqrt{\frac{Kg}{m}} = 2\pi\sqrt{\frac{m}{Kg}}.$$

200. Kinetic Energy of Rotation.—Fig. 276 shows an element of mass dm at a distance r from an axis through O. If this element rotates about the axis with an angular velocity ω, its linear velocity is $r\omega$, and its kinetic energy is

$$dU = \frac{r^2\omega^2 dm}{2g}. \tag{3}$$

The kinetic energy of the entire mass rotating with angular velocity about the axis through O is the integral of Equation (1).

$$U = \frac{\omega^2}{2g}\int r^2 dm; \tag{2}$$

$$U = \text{kinetic energy} = \frac{I\omega^2}{2g}. \qquad \text{Formula XXVIII}$$

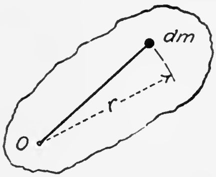

FIG. 276.

If the mass is expressed in pounds and if $g = 32.174$, Formula XXVIII gives the kinetic energy in foot pounds. If the mass is given in kilograms and $g = 9.81$, the formula gives the kinetic energy in kilogram-meters.

Problems

1. A homogeneous solid cylinder 4 feet in diameter and weighing 600 pounds revolves on its axis at the rate of 2 revolutions per second. Find its kinetic energy in foot pounds.

$$\textit{Ans. } U = \frac{600 \times 2 \times 16 \times 9.8696}{2 \times 32.174} = 2945 \text{ ft. lb.}$$

2. A hollow cylinder 6 feet outside diameter and 4 feet inside diameter weighs 800 pounds and revolves on its axis at the rate of 5 revolutions per second. Find its kinetic energy. *Ans.* $U = 79{,}756$ ft. lb.

3. A flywheel has a radius of gyration of 1.8 feet and weighs 600 pounds. Find its kinetic energy when it is making 300 revolutions per minute.

Ans. $U = 29{,}818$ ft. lb.

4. The flywheel of Problem 3 is brought to rest by means of a brake which rubs on the outer surface. The diameter of the wheel is 4 feet. The brake exerts a normal pressure of 100 pounds and the coefficient of friction is 0.24. How many revolutions will the wheel make before it comes to rest?

Ans. 96.3 revolutions.

5. A homogeneous solid cylinder which is 4 feet in diameter and weighs 400 pounds rotates on a frictionless axis. A rope is wound several times around the cylinder and is fastened to it. A constant pull of 120 pounds is applied to the rope while the cylinder makes 5 revolutions. What is the work done by the rope on the cylinder? What is the final velocity of the cylinder at the end of 5 revolutions?

Ans. $U = 120s = 120 \times 4\pi \times 5 = 7540$ ft. lb.;

$$\frac{800\omega^2}{2g} = 7540, \quad \omega = 24.64 \text{ radians per sec.}$$

6. In Problem 5, what is the final velocity of the surface of the cylinder in feet per second. $v = 49.28$ ft. per sec.

7. A *mass* of 120 pounds is hung on the rope of Problem 5, Fig. 277. Find the angular velocity of the cylinder and the linear velocity of the rope after the mass has moved 60 feet.

$$\frac{800\omega^2}{2g} + \frac{120v^2}{2g} = 7200;$$

$$\frac{1280\omega^2}{2g} = 7200;$$

$$\omega = 19.0 \text{ radians per sec; } v = 38.0 \text{ ft. per sec.}$$

8. A homogeneous solid cylinder weighing 200 pounds is 2 feet in diameter. This cylinder is coaxial with a second cylinder which is 6 inches in diameter and weighs 40 pounds. A rope wound on the smaller cylinder supports a mass of 60 pounds. If there is no friction and if the mass starts from rest, what velocity will it acquire while it moves 100 feet?

Ans. $v = 15.16$ ft. per sec.

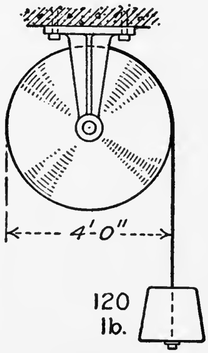

FIG. 277.

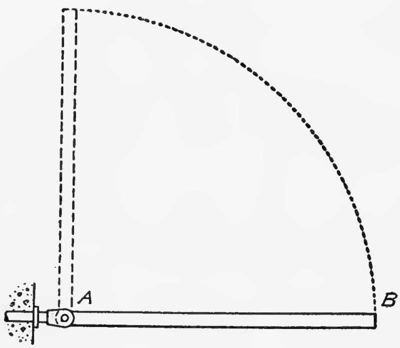

FIG. 278.

9. A uniform rod AB is 6 feet in length and is hinged at A. The rod is placed in the position of unstable equilibrium and then released. Find its velocity when it reaches the horizontal position of Fig. 278.

Ans. Velocity of B is 24.06 ft. per sec.

10. Solve Problem 9 for the velocity when rod is 30 degrees below the horizontal and when the rod is vertical downward.

Ans. 4.91 radians per sec.; 5.67 radians per sec.

201. Rotation and Translation.—In the preceding article, the axis of rotation is stationary and the kinetic energy is the energy of rotation. When a body is rotating about an axis and the axis has a linear motion, the total kinetic energy is the sum of the energy of rotation and the energy of translation.

$$U = \frac{I\omega^2}{2g} + \frac{mv^2}{2g}. \tag{1}$$

Since energy is not a vector quantity, the total energy is arithmetic sum of the two terms.

Figure 279 shows a wheel of radius r which rolls without slipping on a horizontal surface. If the axis were stationary, the velocity of the circumference of the wheel relative to the point B would be $r\omega$. When the wheel does not slip, the velocity of its axis relative to the surface is $v = r\omega$.

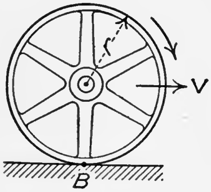

FIG. 279.

Problems

1. A homogeneous solid cylinder is 4 feet in diameter and weighs 240 pounds. It is rolling at a speed of 60 feet per second. Find the kinetic energy of translation, the kinetic energy of rotation, and the total kinetic energy.

Ans. 13,426 ft.-lb.; 6713 ft.-lb.; 20,739 ft.-lb.

2. When a homogeneous cylinder is rolling without slipping on a surface, show that the kinetic energy of rotation is one-half the kinetic energy of translation.

3. A homogeneous solid cylinder rolls down an inclined plane which makes an angle of 20 degrees with the horizontal. What velocity will it acquire while moving 100 feet from rest? What velocity would it acquire if it were to slide down the plane with no friction?

Ans. 38.3 ft. per sec.; 46.9 ft. per sec.

4. In Problem 3, what is the average velocity when the cylinder rolls down the plane? What is the time of descent? *Ans.* $t = 5.22$ sec.

5. The outside diameter of a hollow cylinder is 4 feet and the inside diameter is 3.9 feet. The cylinder rolls 100 feet down a plane which makes an angle of 10 degrees with the horizontal. Find the final velocity of the hollow cylinder. What velocity would a solid cylinder acquire in rolling the same distance down this plane?

Ans. Velocity of hollow cylinder = 23.78 ft. per sec.; velocity of solid cylinder = 27.29 ft. per sec.

6. A sphere which is 8 feet in diameter rolls on two parallel rods which are 1 foot in diameter. The axes of the rods are 5 feet apart. Find the ratio of the angular velocity of the sphere to the linear of velocity of its center.

7. A homogeneous solid sphere rolls on a surface. What is the ratio of its energy of rotation to its energy of translation?

202. Translation and Rotation Reduced to Rotation.—Translation and rotation in the same plane may be regarded as rotation about some axis. Figure 280 shows a cylinder of radius r, which is rolling with velocity v on a plane surface. The angular velocity of the cylinder about its axis is $\omega = \frac{v}{r}\cdot$ The velocity at any point may be found from the resultant of the velocity of the axis of the cylinder and the angular rotation about the axis. The velocity at any point may also be found from the angular motion about the line of contact with the plane. In Fig. 280, the cylinder may be regarded as rotating about an axis through O perpendicular to the plane of the paper. The axis of the cylinder moves a distance vdt during a time interval dt. The angular velocity with respect to the *instantaneous axis* through B is $\omega = \frac{v}{r}$. Since the cylinder is a rigid body, the angular velocity of all parts is the same. All parts of the cylinder may, therefore, be regarded as rotating with angular velocity about the instantaneous axis through B.

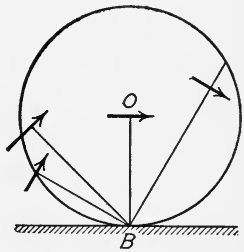

FIG. 280.

The kinetic energy of a body which rolls on a plane surface is

$$U = \frac{I_0\omega^2}{2g} + \frac{mv^2}{2g}. \tag{1}$$

If r is the radius from the axis of the body to the line of contact with the surface, $v = r\omega$ and Equation (1) may be written

$$U = \frac{I_0\omega^2}{2g} + \frac{mr^2\omega^2}{2g}; \tag{2}$$

$$U = \frac{m\omega^2}{2g}(k_0^2 + r^2) = \frac{m^2\omega k^2}{2g} = \frac{I\omega^2}{2g}, \tag{3}$$

in which k_0 is the radius of gyration with respect to the axis through the center of gravity, and k is the radius of gyration with respect to the line of contact with the surface. These equations show that the energy computed from the linear velocity of the center of mass and the angular velocity about that center is the same as the energy computed from the angular velocity about the line of contact.

Problems

1. A homogeneous solid cylinder which is 4 feet in diameter and weighs 600 pounds rolls on a plane surface with a velocity of 20 feet per second. Find the kinetic energy in foot-pounds, regarding the axis as moving 20 feet per second, and the cylinder as rotating on its axis at the rate of 10 radians per second. Also find the kinetic energy, regarding the cylinder as rotating about the axis of contact with the surface with an angular velocity of 10 radians per second. *Ans.* $U = 5594.4$ ft.-lb.

2. A bar of length l is rotating with angular velocity ω about an axis through one end perpendicular to its length. Find its kinetic energy.

3. Solve Problem (2) from the condition that the center of gravity of the bar has a linear velocity of $\frac{l\omega}{2}$ and the bar rotates about the center with an angular velocity of ω radians per sec.

Ans. $U = \frac{ml^2\omega^2}{6g}$.

4. In Fig. 281, find the direction and magnitude of the velocity of each of the points *A*, *B*, *C*, *D*, and *E*. Solve by regarding the axis as moving in a straight line while the points revolve about it.

5. Solve Problem 5 by considering the cylinder as rotating about the line of contact with the plane.

6. A car weighing 200 pounds has a radius of gyration of 2 feet with respect to the vertical axis through its center of gravity. The car moves with a velocity of 40 feet per second on a straight track. If no energy is lost in friction, what is its velocity when it runs on to a curve of 10-foot radius? *Ans.* 39.2 ft. per sec.

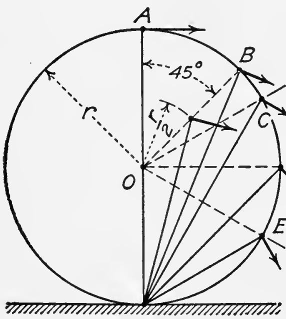

FIG. 281.

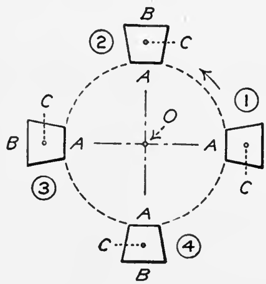

FIG. 282.

Rotation of a body about any axis with an angular velocity of ω radians per second is equivalent to a linear velocity of $r\omega$ feet per second combined with an angular velocity of ω radians per second about a parallel axis through the center of gravity. Figure 282 shows four positions of a body *AB* which rotates about an axis *O* normal to the plane of the paper. The axis through the center of gravity of *AB* passes through *C*. At position 1 at the right of the figure, the point *B* is to the right of *C*. At the second position, *B* is above *C*. At the third position, *B* is to the left of *C*. At the last position, *B* is below *C*. It is evident, therefore, that when the body makes one revolution about *O*, it makes one revolution about *C*.

203. Summary.—Angular displacement is measured in radians. Angular velocity is measured in radians per second. When the velocity is constant, the displacement is the product of the velocity multiplied by the time. When the velocity varies, the displace ment is the product of the average velocity multiplied by the time.

If r is the distance from a point to the axis of rotation, the *linear* velocity of the point due to an angular velocity of ω radians about the axis is given by

$$v = r\omega. \qquad \text{Formula XXVII}$$

This linear velocity is normal to the direction of r.

If the axis of rotation is moving with a linear velocity, the resultant linear velocity of any point is obtained by combining the linear velocity of rotation with the linear velocity of the axis. These velocities are added as vectors.

The kinetic energy of a rotating body is

$$U = \frac{I\omega^2}{2g}. \qquad \text{Formula XXVIII}$$

This expression for kinetic energy corresponds with the expression for the kinetic energy of a body which has a linear velocity. Angular velocity replaces linear velocity, and moment of inertia replaces mass or inertia.

If the axis of rotation is moving with a linear velocity, the total energy is the sum of the energy of rotation and the energy of translation. Energy is a scalar quantity; kinetic energy is always positive.

CHAPTER XXII

ACCELERATION TOWARD THE CENTER

204. Components of Acceleration.—In Fig. 283, a point at A_1 in the circumference of a circle of radius r is moving with a linear velocity v_1. When the point reaches the position A_2, its velocity is v_2. The direction and magnitude of the velocity at A_1 is given by the tangent A_1C; the direction and magnitude of the velocity at A_2 is given by the tangent A_2D. Figure 283, II, is the vector diagram of the velocities with a common initial point A. The vector CD of Fig. 283, II, represents the change of velocity while the body moves from A_1 to A_2. This increment

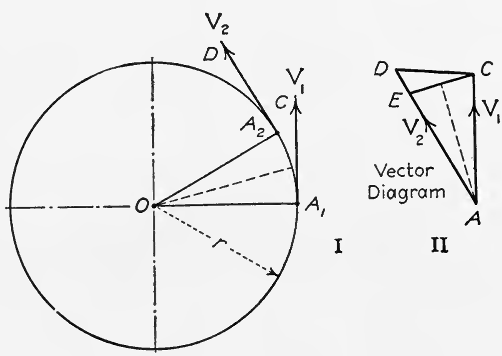

FIG. 283.

of velocity may be resolved into two components CE and ED. The length AE is made equal to AC. The component ED represents the increment which is due to change in the magnitude of the velocity; the component CE represents the increment which is due to change in the direction. Since AEC is an isosceles triangle, the direction of CE is normal to the line which bisects the angle CAE, and is parallel to the line which bisects the line A_1OA_2 of Fig. 283, I

205. Acceleration Toward the Center.—If v is the magnitude of the velocity at A_1 and if the distance from A_1 to A_2 is infinitesimal, the increment CE is

$$dv = v d\theta, \tag{1}$$

in which $d\theta$ is the change in the direction of the velocity and dv is the increment which is due to the change in direction. Since the direction of the increment is perpendicular to the line which makes an angle $\frac{d\theta}{2}$ with the direction of v_1, the change of velocity is toward the center of the circle of Fig. 283, I.

The linear acceleration toward the center is

$$a = \frac{dv}{dt} = \frac{v d\theta}{dt}. \tag{2}$$

Since $A_1A_2 = r\, d\theta = vdt,$

$$\frac{d\theta}{dt} = \frac{v}{r}.$$

$$a = \frac{v^2}{r}. \qquad \text{Formula XX}$$

Since $v = r\omega;$

$$a = r\omega^2. \tag{3}$$

Formula XX has already been derived in Art. 162 for motion in a circular path with uniform speed.

Problems

1. A body is whirled in a circular path 6 feet in diameter with a speed of 15 feet per second. Find its acceleration toward the center due to change of direction. *Ans.* $a = 75$ ft. per sec. per sec.

2. A wheel 4 feet in diameter is making 5 revolutions per second. What is the acceleration toward the center of a point on the rim of the wheel?

Ans. $a = 2 \times 10^2 \times \pi^2 = 1974$ ft. per sec. per sec.

3. A wire is bent into the form of an ellipse the equation of which is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

A body moves on this wire with a constant speed v. Find the acceleration of the body when it is at the end of the major axis. *Ans.* $a = \frac{v^2 a}{b^2}.$

Problem 3 may be solved by substituting the radius of curvature of the ellipse in Formula XX. Another solution, which does not depend upon Formula XX, is given below. Differentiating the equation of the ellipse with respect to t,

$$\frac{x}{a^2}\frac{dx}{dt} + \frac{y}{b^2}\frac{dy}{dt} = 0 \tag{4}$$

When $x = a,$ $y = 0,$ Substitution in E uation (4) shows that $\frac{dx}{dt} = 0$ when $x = a.$ Since the X component of the velocity is zero at this position, the Y component must equal v; $\frac{dy}{dt} = v$ when $x = a$. The motion at the point A of Fig. 284 is vertically upward.

Differentiating Equation (4) with respect to t,

$$\frac{x}{a^2}\frac{d^2x}{dt^2} + \frac{1}{a^2}\left(\frac{dx}{dt}\right)^2 + \frac{y}{b^2}\frac{d^2y}{dt^2} + \frac{1}{b^2}\left(\frac{dy}{dt}\right)^2 = 0. \quad (5)$$

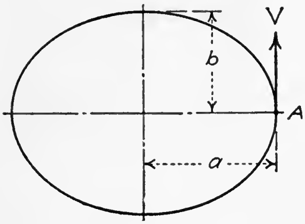

FIG. 284.

From the vector triangle of velocities,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = v^2 \quad (6)$$

Differentiating Equation (6) with v constant,

$$\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} = 0. \quad (7)$$

When $x = a$, $\frac{dx}{dt} = 0$. Substitution in Equation (7) show that $\frac{d^2y}{dt^2}$ is zero; hence the acceleration has no vertical component when $x = a$. Substituting the conditions that the horizontal component of the velocity and the vertical component of the acceleration are each zero when $x = a$, Equation (5) becomes

$$\frac{x}{a^2}\frac{d^2x}{dt^2} + \frac{1}{b^2}\left(\frac{dy}{dt}\right)^2 = 0; \quad (8)$$

$$\frac{d^2x}{dt^2} = \frac{a}{b^2}\left(\frac{dy}{dt}\right)^2 = \frac{av^2}{b^2}. \quad (9)$$

If $a = b$, the ellipse is a circle and Equation (9) reduces to Formula XX.

206. Centrifugal Force.—When a particle is rotating about an axis, the acceleration toward the axis which results from the change of direction is

$$a = \frac{v^2}{r} = r\omega^2.$$

Since the force which produces acceleration is $\frac{ma}{g}$, the force which produces this acceleration in a particle of mass dm is $\frac{dmv^2}{rg}$ or $\frac{dmr\omega^2}{g}$. This is called the centrifugal force. The force on the particle is toward the axis; it is, therefore, centripetal force. The force which the rotating particle exerts on the body or surface which holds it to a curved path is directed away from the axis; it is, therefore, centrifugal.

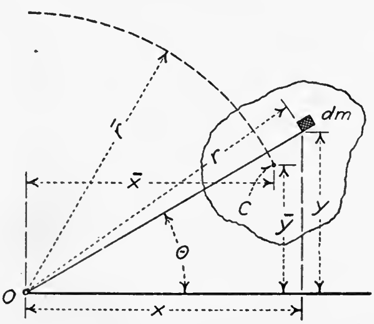

FIG. 285.

All parts of a small particle are at the same distance from the axis. When a body of considerable dimensions is rotating about an axis, it is necessary to find an expression for the resultant centrifugal force exerted by the entire body. Figure 285 represents a body which is rotating about an axis at O normal to the plane of the paper. The element of mass dm at a distance r from the axis is subjected to a pull toward the axis. The magnitude of this pull is

$$dP = \frac{r\omega^2 dm}{g} \tag{1}$$

To find the resultant pull on all the elements of mass, this force is resolved into components parallel to the coordinate axes. The resolution parallel to the X axis is

$$dP_x = \frac{\omega^2}{g} r \cos\theta \, dm = \frac{\omega^2}{g} xdm; \tag{2}$$

$$P_x = \frac{\omega^2}{g}\int xdm = \frac{\omega^2 m\bar{x}}{g}. \tag{3}$$

In a similar way, the component parallel to the Y axis is found to be

$$P_y = \frac{\omega^2 m\bar{y}}{g}. \tag{4}$$

The resultant force is

$$P = \frac{m\bar{r}\omega^2}{g} = \frac{mv^2}{g\bar{r}}, \qquad \text{Formula XXIX}$$

in which $\bar{r}$ is the distance of the center of gravity of the mass from the axis, and v is the linear velocity of the center of gravity.

Problems

1. A mass of 4 pounds is whirled in a horizontal plane at the end of a rod 4 feet in length. The velocity of the center of gravity of the body is 20 feet per second. What tension does the body exert on the rod?

Ans. $P = 12.43$ lb.

2. The body of Problem 1 makes 2 revolutions per second about a vertical axis. Find the tension in the rod. Use the formula in terms of the angular velocity.

Ans. $P = 78.53$ lb.

3. A mass of 5 pounds is whirled in a vertical plane at the end of a cord 5 feet in length. Find the tension at the top and bottom of its path if the velocity at each position is 24 feet per second.

Ans. 12.90 lb. at the top; 22.90 lb. at the bottom.

4. In Problem 3, the velocity at the top is 24 feet per second. If no force acts except gravity, find the velocity at the bottom and the tension in the cord.

5. A pail of water is whirled in a vertical plane about a horizontal axis.

The distance from the axis to the surface of the water is 3 feet. Find the minimum velocity.

6. In a "Loop the Loop," the radius of curvature at the top is 20 feet. How much must the starting point be elevated above the top of the loop if no allowance is made for loss of energy which is due to friction, or for the kinetic energy of rotation?

7. In a "Loop the Loop," the starting point is 40 feet above the lowest point. The radius of curvature at the lowest point is 40 feet. What will be the weight on a spring balance of a 200-pound body when passing the lowest point? *Ans.* $P = 600$ lb.

8. A boy on a bicycle makes a turn of 40 feet radius at a speed of 20 feet per second. How much must he lean from the vertical? *Ans.* 17° 16′.

9. If the boy of Problem 8 makes the turn on a level street, what must be be the minimum value of the coefficient of friction? *Ans.* $f = 0.311$.

10. If the distance between rails of a track is 56.5 inches, how much must the outer rail be elevated for a curve of 800 feet radius in order that the resultant force may be normal at a speed of 30 miles per hour? *Ans.* 4.25 inches.

11. A dynamo armature weighing 400 pounds runs at 1200 revolutions per minute. If its center of gravity is 0.03 inch from its axis, what is the maximum vertical pull upward on the bearings, and what is the maximum horizontal force on the bearings? *Ans.* 90.8 lb. 490.8 lb.

207. Static Balance.—The armature of Problem 11 of the preceding article is "out of balance." The resultant pressure downwards when the center of gravity is below the axis is 890 pounds. The resultant pressure upward, when the center of gravity is above the axis, is 90 pounds. The increased pressure which is due to centrifugal force bends the shaft. This deflection of the shaft increases the distance of the center of gravity from the axis of rotation and still further increases the centrifugal force. If the time of vibration of the shaft should coincide with the time of one revolution, the deflection and stress would be increased still more. Even if the condition of *resonance* does not exist, the additional stresses which are due to the centrifugal force are dangerous and destructive. It is evident, therefore that rapidy running machinery must be accurately balanced so that the center of gravity of the rotating part may coincide with its axis of rotation.

In Problem 11, the moment of the mass about the axis is $400 \times 0.03 = 12$ inch pounds. This moment may be balanced by a mass of 6 pounds at 2 inches from the axis or a mass 4 pounds at 3 inches from the axis.

The static balance of an armature or other body designed to rotate at high speed is frequently determined by placing the axle on a pair of horizontal rails. The axle rolls on these rails until the center of gravity is vertically below the axis. Additional load is then placed directly above the axle to secure a balance. When the axle is in equilibrium at any position, the resultant center of gravity is known to be on the axis. The body is then in *static balance.*

208. Running Balance.—A shaft may be in perfect static balance and yet when running, be decidedly out of balance. For perfect running balance, the shaft and the bodies which it carries must be symmetrical with respect to the axis. In Fig.

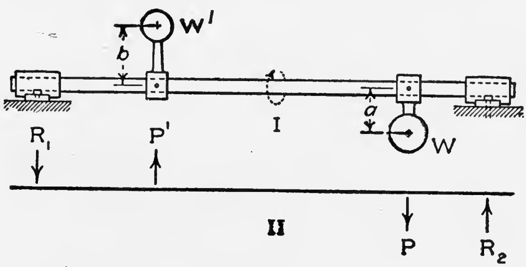

FIG. 286.

286, I, the mass W is at a distance a from the axis. A second mass W' at a distance b from the axis on the opposite side will produce a static balance if

$$W'b = Wa.$$

In Fig. 286, II, P' represents the centrifugal force which is due to W', and P in the opposite direction represents the equal centrifugal force which is due to W. Since these forces are not along the same line, they produce bending moments in the shaft and varying reactions in the bearings.

Example

In Problem 11 of the preceding article, the axle is 60 inches long from center to center of bearings. The center of mass of the armature is 32 inches from the left bearing. Static balance is secured by the addition of 2.4 pounds of lead inside a collar located 10 inches from the right bearing. The center of gravity of the lead is 5 inches from the axis. Find the reactions of the bearings when the armature is making 1200 revolutions per minute.

The centrifugal force of the mass of the armature is 490.8 pounds. The centrifugal force of the 2.4 pounds of lead is also 490.8 pounds. In Fig. 287, these centrifugal forces are horizontal. The horizontal reactions of the bearings are H_1 and H_2. By moments about the vertical lines through the bearings, H_1 is found to be 147.2 pounds in one direction and H_2 is found to be an equal force in the opposite direction. When the center of gravity is below the axis, the centrifugal force adds 147.2 pounds to the

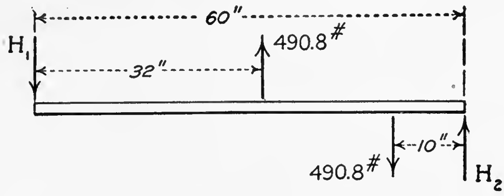

Fig. 287.

load on the left bearing and subtracts the same amount from the load on the right bearing.

Problems

1. Find the reactions of the bearings of the above example when the center of gravity of the armature is directly below the axis.

Ans. 334.3 lb.; 65.8 lb.

2. Solve Problem 1 when the center of gravity of the armature is directly above the axis of the shaft.

209. Ball Governor.—In Fig. 288, AB is shaft which rotates about a vertical axis, and C is a heavy ball connected to the shaft by a rod. The rod is attached to the shaft at B by means of a hinge which permits the rod and the ball C to rotate in a vertical plane. As the vertical shaft rotates, the centrifugal force in C develops a moment which tends to lift BC toward a horizontal position.

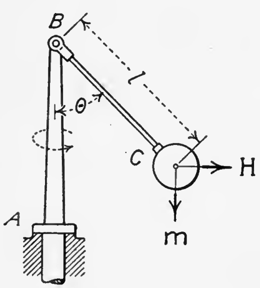

Fig. 288.

If θ is the angle which BC makes with the vertical, if l is the length of BC from the hinge to the center of gravity, and if m is the combined mass of the ball and the rod, the centrifugal force is

$$H = \frac{ml \sin \theta \omega^2}{g}. \tag{1}$$

The moment arm of this horizontal centrifugal force about the hinge at B is $l \cos \theta$ while the moment arm of the weight is $l \sin \theta$. Equating the moment of the centrifugal force to the moment of the weight,

$$ml \sin \theta = \frac{ml \sin \theta \omega^2 l \cos \theta}{g} \tag{2}$$

$$\cos \theta = \frac{g}{l\omega^2} \tag{3}$$

Problems

1. The length of a governor arm is 12 inches. At what speed will the arm stand at 60 degrees with the vertical?

Ans. 8.02 radians per sec.; 76.6 rev. per min.

2. In Problem 1, what is the value of θ when the speed is 50 revolutions per minute? *Ans.* $\theta = 0$.

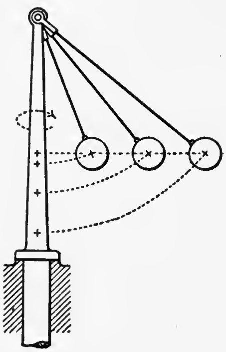

FIG. 289.

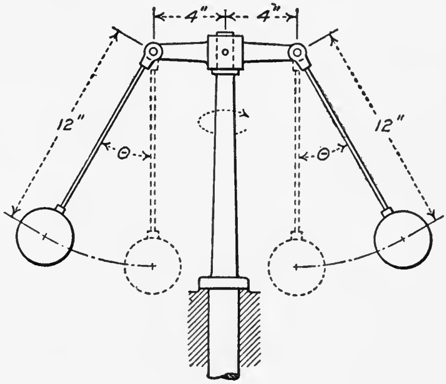

FIG. 290.

3. Figure 289 shows several arms of different lengths which are attached to the same hinge. Show that the center of gravity of each will fall on the same horizontal line, provided the angular velocity is sufficient to make all of them leave the vertical position.

4. In Problem 1, find the angle for angular velocities of 6, 7, 8, 9, and 10 radians per second. *Ans.* 26° 40′, 48° 58′, 59° 49′, 66° 36′, and 71° 14′.

5. In Problem 1, find the angular velocity for angles of 30°, 40°, 50°, 60°, and 70°.

Ans. 6.10, 6.48, 7.07, 8.02, and 9.70 radians per sec.

6. In Fig. 290, find the angular velocity for angles of 30°, 40°, 50°, 60°, and 70°.

Ans. 4.72, 5.26, 5.90, 6.82, and 8.33 radians per sec.

7. Figure 291 shows a crossed-arm governor such as is used on some chronographs. Find the expression for the angular velocity in terms of the angle θ, the length a, and the length l.

$$Ans. \quad \omega^2 = \frac{g \tan \theta}{l \sin \theta - a}.$$

If l in Fig. 291 is 12 inches and a is 4 inches, find the value of the angular velocity for angles of 45°, 50°, 60°, and 70°.

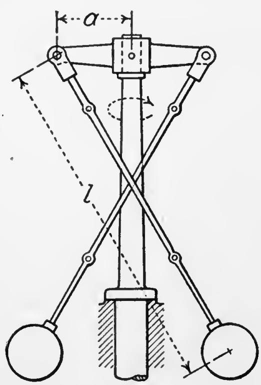

FIG. 291.

Ans. 9.28, 9.41, 10.23, and 12.07 radians per sec.

The form of governor shown in Fig. 291 is extremely sensitive when θ is about 45 degrees. A change of angular velocity of less than 2 per cent causes a change in the angle from 45° to 50°.

210. Loaded Governor.—Figure 292 shows a loaded governor. In this figure, the length of the links which connect the load with the arms is equal to the distance from the top of the arm to the connection with the link. The link and the arm, therefore, make equal angles with the vertical. The centrifugal force is

$$H = \frac{ml \sin \theta\omega^2}{g}. \tag{1}$$

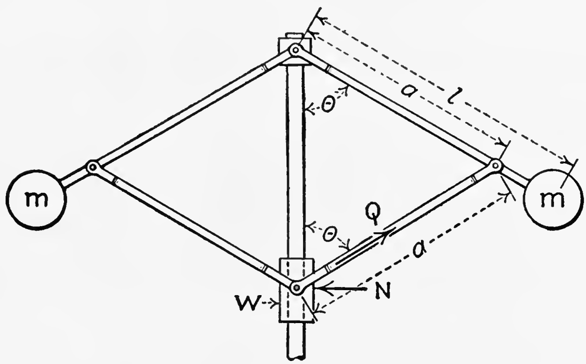

FIG. 292.

If Q is the tension in one link and W is the weight lifted by both links,

$$Q \cos \theta = \frac{W}{2}. \tag{2}$$

The horizontal component of Q is

$$N = Q \sin \theta = \frac{W}{2} \tan \theta. \tag{3}$$

Consider the right arm and the right link as a system in equilibrium, and take moments about the top of the arm. The force from the weight to the link may be resolved into a vertical component downward and a horizontal component N directed toward the left. The moment arm of the vertical component is zero. The moment arm of the horizontal component is $2a \cos \theta$.

$$ml \sin \theta + \frac{W}{2} 2\, a \tan \theta \cos \theta = \frac{ml \sin \theta\omega^2}{g} l \cos \theta; \tag{4}$$

$$ml + Wa = \frac{ml^2\omega^2}{g} \cos \theta. \tag{5}$$

Problems

1. In Fig. 292, the length l is 20 inches and the length a is 12 inches. The load W is 12 pounds and the weight of each arm is 10 pounds. Find the speed when the arms make an angle of 45 degrees with the vertical.

Ans. 6.85 radians per sec.

2. What is the angular velocity required to lift the weight of Problem 1 a distance of 2 inches from the position at which the arms make an angle of 45 degrees with the horizontal? *Ans.* 7.30 radians per sec.

211. Flywheel Stresses.—When a wheel is rotating, every part of its mass exerts a pull due to the centrifugal force. In calculating the stresses, the rim is often considered separately and the mass of the spokes is neglected. The force on a rim is similar to the force exerted on a cylindrical drum from the pressure of confined liquid or gas. The centrifugal force on an element of mass of the rim is $\frac{r\omega^2}{g}$. The rim of Fig. 293 may be considered as made up of circular hoops each of which is 1 inch wide and t inches thick. If w is the weight of a rod of the material 1 inch square and 1 foot long, the centrifugal force per foot of the hoop is $\frac{wtr\omega^2}{g}$. To find the total force on one half of the hoop which tends to rupture it at the sections A and B, the forces on this half are resolved normal to the diameter through A and B. The sum of all the components is equal to a force of $\frac{wtr\omega^2}{g}$ per foot of length of the diameter.

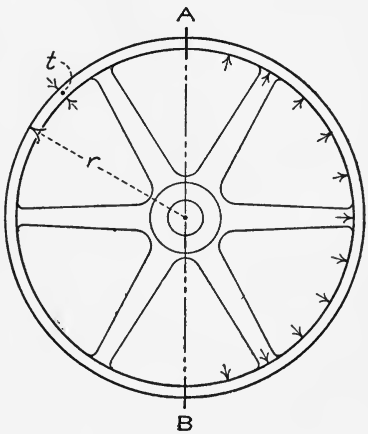

FIG. 293.

$$\text{Total pull} = \frac{wtr\omega^2}{g}2r = \frac{2wtr^2\omega^2}{g} \tag{1}$$

This pull is resisted by the tensile strength of the material at A and B. The total cross section is $2t$. If S_t is the unit tensile stress,

$$2tS_t = \frac{2wtr^2\omega^2}{g}; \tag{2}$$

$$S_t = \frac{wr^2\omega^2}{g} = \frac{wv^2}{g}. \tag{3}$$

The weight of a rod of steel 1 inch square and 1 foot long is 3.4 pounds. The weight of a rod of cast iron is about 3.2 pounds.

Problems

1. The rim of a steel wheel is moving with a speed of 80 feet per second. Find the tension due to the centrifugal force. *Ans.* $S_t = 677$ lb./in.^{2}

2. A steel wheel 6 feet in diameter is making 300 revolutions per minute. Find the tension in the rim due to rotation. *Ans.* $S_t = 939$ lb./in.^{2}

3. If the density of cast iron is 90 per cent the density of steel, and if the allowable tensile stress is 2000 pounds per square inch, what is the maximum allowable rim velocity for cast iron wheels?

Equations (2) and (3) may be derived by calculating the centrifugal force on one half of the rim. If the breadth is b and the thickness is t, the volume of one-half the rim is $bt\pi r$. If r is in feet and b and t are expressed in inches, the mass is $wbt\pi r$. The centrifugal force is $\frac{wbt r\pi \bar{r}\omega^2}{g}$, in which $\bar{r}$ is the distance of the center of gravity of the semicircular rim from the axis. If the thickness is small, $\bar{r} = \frac{2r}{\pi}$. Substituting this value of $\bar{r}$ and equating to the tensile stress.

$$\frac{2wbtr^2\omega^2}{g} = 2btS_t;$$

$$S_t = \frac{wr^2\omega^2}{g} = \frac{wv^2}{g}.$$

If w' is the density in pounds per cubic inch,

$$S_t = \frac{12w'v^2}{g}$$

212. Summary

A rotating body may have two accelerations. One of these accelerations is due to the change of direction. This acceleration is directed toward the axis of rotation. Its magnitude is

$$a_c = \frac{v^2}{r} = r\omega^2.$$

The centrifugal force exerted by a particle of m pounds mass is

$$P = \frac{mv^2}{rg} = \frac{mr\omega^2}{g}.$$

The centrifugal force exerted by a body of some magnitude is

$$P = \frac{mv^2}{\bar{r}\,g} = \frac{m\bar{r}\omega^2}{g}. \qquad \text{Formula XXIX}$$

in which $\bar{r}$ is the distance of the center of gravity of the mass from the axis of rotation and v is the linear velocity of the center of gravity.

A rotating body is in *static* balance when the center of gravity coincides with the axis of revolution. A body is in *running* balance when the center of gravity of every small portion between two planes normal to the axis falls on the axis.

In a Watt or ball governor, the arms take positions of equilibrium under the action of the horizontal centrifugal force which varies as the square of the angular velocity and as the distance from the axis, and the constant vertical weights of the arms and the additional loads.

Allowance must be made for stresses which are due to centrifugal force in high speed machinery. Since centrifugal force varies as the square of the angular velocity, a relatively small increase of speed may become a source of danger.

CHAPTER XXIII

ANGULAR ACCELERATION

213. Angular Acceleration.—The rate of change of angular velocity is the angular acceleration.

$$\alpha = \frac{\omega_2 - \omega_1}{t_2 - t_1}, \tag{1}$$

in which α is the angular acceleration. Angular acceleration is expressed in radians per second per second. For small intervals of time, the angular acceleration is

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}. \tag{2}$$

Problems

1. A fly wheel is making 2 revolutions per second, after an interval of 4 seconds, it is making 5 revolutions per second. Find the angular acceleration. By means of the average velocity, find the number of revolutions during the 4-second interval.

Ans. α = 4.712 radians per sec. per sec.; 14 revolutions.

2. A point on the rim of a wheel 6 feet in diameter is moving with a velocity of 60 feet per second. After 2 seconds, it is moving with a velocity of 90 feet per second. Find the angular acceleration of the wheel.

Ans. 5 radians per sec. per sec.

3. A pencil held for 0.1 second against the rim of a wheel which is 8 feet in diameter makes a mark 18 inches long. Two seconds later the pencil is held against the wheel for the same time and makes a mark 30 inches long. Find the angular acceleration of the wheel.

Ans. α = 1.25 radians per sec. per sec.

214. Displacement with Constant Acceleration.—The expressions for angular velocity and displacement when the acceleration is constant are similar to those for linear displacement and velocity when the linear acceleration is constant.

$$\omega = \omega_0 + \alpha t \tag{1}$$

$$\theta = \omega_0 t + \frac{\alpha t^2}{2} \tag{2}$$

Equation (1) may be derived from the definition of acceleration.

Equation (2) may be derived by multiplying the average angular velocity by the time. Both equations may be derived by integration.

Problems

1. A wheel has an angular acceleration of 2 revolutions per second per second. Its initial velocity is 3 revolutions per second. What will be its velocity after 4 seconds? *Ans.* 11 revolutions per second.

2. How many revolutions will the wheel of Problem 1 make during the interval of 4 seconds. Solve by means of the average velocity. *Ans.* 26 revolutions.

3. A wheel is making 1600 revolutions per minute. After an interval of 20 seconds, it is making 600 revolutions per minute. Find its acceleration in radians per second per second. *Ans.* $\alpha = -5.24$ radians per sec. per sec.

4. How many revolutions does the wheel of Problem 3 make during the interval of 20 seconds? *Ans.* 367 rev.

215. Acceleration and Torque.—In Fig. 294, a mass dm rotates about an axis at O. The angular acceleration is α and the distance of the element from the axis is r. The linear acceleration of the element of mass is $a = r\alpha$. The force which must be applied to produce this linear acceleration is

$$dP = \frac{r\alpha dm}{g}. \qquad (1)$$

The moment of this force with respect to the axis through O is

$$dT = \frac{r^2\alpha dm}{g} = \frac{\alpha}{g}r^2 dm. \qquad (2)$$

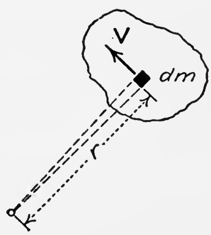

Fig. 294.

The moment required to produce an angular acceleration in a mass made up of elements dm is

$$T = \frac{\alpha}{g}\int r^2 dm = \frac{I\alpha}{g}. \qquad \text{Formula XXX}$$

The moment with respect to an axis of a rotating body is called *torque*. Torque is represented algebraically by the letter T.

Formula XXX for angular acceleration is similar to Formula XXII for linear acceleration. Force in Formula XXII is replaced by moment of force, and mass (or inertia) is replaced by moment of inertia.

If force is expressed in absolute units, Formula XXX becomes

$$T = I\alpha \qquad (3)$$

Equation (3) is sometimes given as the physical definition of moment of inertia. The moment of inertia of a body is numerically equal to the moment of the force (in absolute units) which gives to the body an acceleration of 1 radian per second per second.

Problems

1. A homogeneous solid cylinder is 4 feet in diameter and weighs 240 pounds. A rope to which a force of 60 pounds is applied is wound several times around the cylinder. What is the torque? What is the angular acceleration? What is the angular velocity after 4 seconds and what angle is turned through in 4 seconds if the cylinder starts from rest?

Ans. T = 120 ft.-lb.; α = 8.044 radians per sec. per sec.; ω = 32.176 radiance per sec.; θ = 64.35 radians.

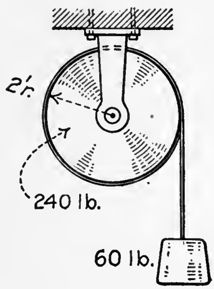

FIG. 295.

2. Figure 295 shows the cylinder of Problem 1 with a *mass* of 60 pounds hung on the free end of the rope. Find the acceleration, the tension in the rope, and the angular velocity after 4 seconds.

If P is the tension in the rope, the effective force on the mass of 60 pounds is $60 - P$. When the 60-pound mass is considered as a free body with a linear acceleration a, Formula XXII gives

$$60 - P = \frac{60a}{g}. \qquad (4)$$

The torque on the cylinder is $2P$ and Formula XXX gives

$$2P = \frac{480\alpha}{g}. \qquad (5)$$

Substituting $a = 2\alpha$ in Equation (4) and eliminating P between Equations (4) and (5)

$$\alpha = \frac{g}{6} = 5.362 \text{ radians per sec. per sec.;}$$

$$P = 40 \text{ lb.}$$

One-third of the weight of the 60-pound mass is used to accelerate the mass itself and the remaining 40 pounds accelerates the cylinder.

3. In Problem 2 find the distance traversed by the mass of 60 pounds during 4 seconds. Find the work done by gravity and equate with the total kinetic energy.

Ans. Total work = 5147.8 ft.-lb.

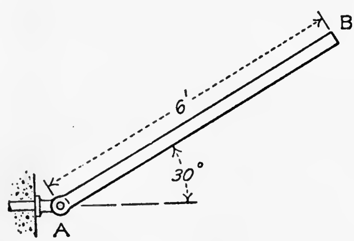

FIG. 296.

4. A homogeneous solid cylinder rolls without slipping down 20-degree inclined plane. Find the acceleration. Solve by considering the cylinder as rotating about its line of contact with the plane.

$$\textit{Ans.} \quad \alpha = \frac{2g \sin 20°}{3r} = \frac{7.39}{r} \text{radians per sec. per sec.;}$$

$$a = 7.34 \text{ ft. per sec. per sec.}$$

5. A uniform bar AB, Fig. 296, 6 feet long and weighing 12 pounds, is hinged at A. What is the angular acceleration when the bar is in the position at which it makes an angle of 30 degrees with the horizontal?

Ans. $\alpha = 3.483$ radians per sec. per sec.

6. Solve Problem 5 for the horizontal position of the bar.

Ans. $\alpha = 8.044$ radians per sec. per sec.

216. Equivalent Mass.—Problem 2 of the preceding article may be solved by first finding the mass which would have the same velocity as the rope and the same moment of inertia as the cylinder. If all the mass of the cylinder were concentrated at the surface, it would move with the linear velocity of the rope. The moment of inertia of a solid cylinder is

$$I = \frac{mr^2}{2};$$

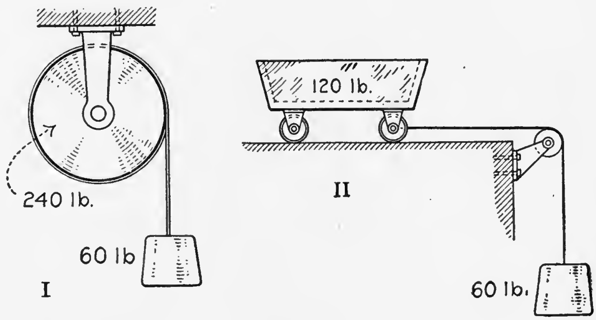

FIG. 297.

a mass $\frac{m}{2}$, concentrated at the surface in the form of a very thin hollow cylinder of radius r, has the same moment of inertia as the actual solid cylinder of mass m. The problem of accelerating a solid cylinder of mass m by means of a rope to which a mass m' is attached is equivalent to that of accelerating a thin hollow cylinder of mass $\frac{m}{2}$. The problem of accelerating a thin hollow cylinder is equivalent to the problem of Fig. 297, II. The mass of 240 pounds of Problem 2 of Art. 215 may be replaced by a mass of 120 pounds on a horizontal plane as in Fig. 297, II and the problem may be solved as a problem of linear acceleration. Since the effective force on the system is 60 pounds and the total equivalent mass is 180 pounds, it is evident that the linear acceleration is $g/3$ and that one-third of the force of 60 pounds is required to accelerate its own mass.

Problems

1. A wheel is 6 feet in diameter and weighs 400 pounds. Its radius of gyration is 2.4 feet. What mass concentrated at the rim will have the same moment of inertia? *Ans.* 256 lb.

2. A wheel is 5 feet in diameter and weighs 600 pounds. Its radius of gyration is 2 feet. A ½-inch rope is wound around the wheel and a mass of 80 pounds is applied to the free end. Find the acceleration and the tension on the rope.

3. The wheel of Problem 2 turns on a 4-inch axle. The coefficient of friction is 0.04. Find the acceleration and the tension on the rope.

4. A cylinder 4 feet in diameter and weighing 200 pounds is coaxial with an axle of the same material which is 6 inches in diameter. The part of the axle which projects from the cylinder weighs 100 pounds. A 1-inch rope is wound round the axle and a mass of 200 pounds is hung on it. If there is no friction, what is the acceleration of the rope and what is the tension in it?

217. Reaction of Supports.—Figure 298 represents a body which is free to rotate in the plane of the paper about a hinge at O. The center of mass of the body is located at C at a distance r from the hinge. A force P parallel to the plane of the paper intersects the plane through the hinge and the center of mass at a distance r' from O. The angle between the force P and the line OC is ϕ. It is desired to find the hinge reaction when the body is rotating about it with accelerated angular velocity.

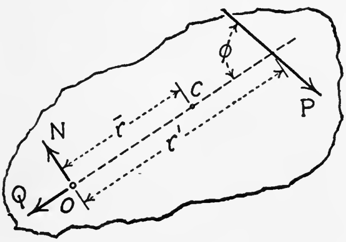

FIG. 298.

The hinge reaction may be resolved into two components. One component **N** is normal to OC; the other component **Q** is in the direction of the line from C to O.

If the body rotates about O with angular velocity ω, the centrifugal force which it exerts on the hinge is in the direction of OC and is equal to $\frac{m\bar{r}\omega^2}{g}$. The component of **P** in the direction of OC is $P \cos \phi$. Resolving parallel to CO,

$$Q = \frac{m\bar{r}\omega^2}{g} + P \cos \phi. \tag{1}$$

The component **Q** depends upon the angular velocity and the applied forces. It is independent of the angular acceleration.

If the center of mass at C has a linear acceleration a, the total effective force on the body in the direction of this acceleration (which is normal to the direction of OC) is $\frac{ma}{g}$. Since the linear acceleration is $r\alpha$, in which α is the angular acceleration about the hinge, this force is $\frac{m\bar{r}\alpha}{g}$. Resolving along an axis in the plane of the paper normal to OC.

$$P \sin \phi - N = \frac{mr\alpha}{g}; \tag{2}$$

$$N = P \sin \phi - \frac{m\bar{r}\alpha}{g}. \tag{3}$$

In Equation (3) the positive direction of **N** is opposite the direction of the linear acceleration of the center of mass.

The torque about the hinge is $Pr' \sin \phi$. From Formula XXX, the angular acceleration is

$$\alpha = \frac{Tg}{I} = \frac{Pr'g \sin \phi}{I}.$$

The force **P** may be resultant of several forces. If any of the forces have components parallel to the axis of the hinge, these components produce no torque about the hinge. The force **P** is the resultant of all the components in planes which are normal to the axis of the hinge.

Example

A uniform bar of length l and mass m is hinged at one end. Find the hinge reaction when the bar is horizontal.

The force which produces angular acceleration is the weight of m pounds vertically downward at the center of mass which is at a distance of $\frac{l}{2}$ from the hinge. (In this example $\bar{r} = r' = \frac{l}{2}$.)

$$T = \frac{ml}{2}; I = \frac{ml^2}{3};$$

$$\alpha = \frac{3g}{2l};$$

$$a = \bar{r}\alpha = \frac{l}{2} \times \frac{3g}{2l} = \frac{3g}{4}.$$

$$N = m - \frac{ma}{g} = m - \frac{m}{g} \times \frac{3g}{4} = \frac{m}{4}.$$

When the bar is horizontal, the vertical reaction is one-fourth the weight.

To find the horizontal component of the hinge reaction, it is necessary to know the angular velocity. As an additional condition of the example, it will be assumed that the bar was stationary in the vertical position and fell to the horizontal position of Fig. 299 with no loss of energy. Equating the work of gravity to the kinetic energy of rotation,

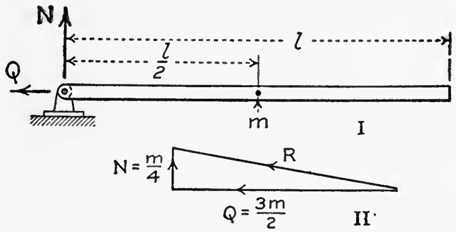

Fig. 299.

$$\frac{ml}{2} = \frac{I\omega^2}{2g};$$

$$\frac{\omega^2}{g} = \frac{ml}{\frac{ml^2}{3}} = \frac{3}{l}$$

The centrifugal force is

$$\frac{m\bar{r}\omega^2}{g} = \frac{ml}{2} \times \frac{3}{l} = \frac{3m}{2}.$$

The force on the hinge is toward the center of mass of the bar. The hinge reaction is horizontal toward the left. The direction and magnitude of the resultant hinge reaction are shown in Fig. 299, II.

Problems

1. Solve the Example above when the bar is 30 degrees above the horizontal as shown in Fig. 300.

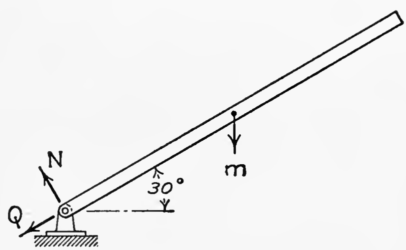

Fig. 300.

Ans. $Q = \frac{m}{4}$; $N = \frac{m}{4}\cos 30° = 0.2163m.$

2. Find the reactions in Problem 1 of Art. 215 if the force of 60 pounds is vertically downward. *Ans.* 300 lb. vertically upward.

3. Find the reactions of the supports for Problem 2 of Art. 215.

218. Reactions by Experiment.—Figure 301 shows a method of measuring the reaction of the bearings which support a cylinder. The frame AB which supports the bearings is mounted on knife-edges at O. The distance from the line of the knife-edges to the axis of the cylinder is equal to the sum of the radius of the cylinder and the radius of the cord which drives it. The line of the resultant force in the cord intersects the line of the knife-edges. If the cylinder is held so that it can not revolve on its axis, a weight hung on the cord has no effect on the equilibrium of the frame. If the cylinder is free to rotate, there is an additional force at the axis equal to that part of the tension in the cord which goes to produce acceleration. If there is friction at the bearings, that part of the tension in the cord which is required to overcome the friction has no effect on the balance. A force P applied to the cylinder by the cord is transmitted to the frame at the bearing. The moment of this force about the knife-edge at a distance r from the axis is Pr in a counter-clockwise direction. If the cylinder is fixed to the frame, the force in the cord exerts a torque Pr upon the frame at the bearing.

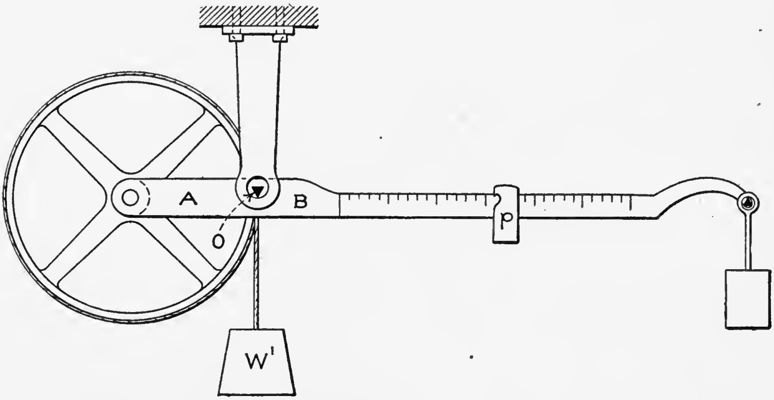

FIG. 301.

This torque is clockwise and exactly balances the moment of the downward force. Expressed in another way, the moment arm of the cord about the line of the knife-edge is zero. If the frame and cylinder are not rigidly connected, that part of the force which is required to overcome the friction may be regarded as acting on a rigid body. The friction at the bearing exerts a torque on the frame which is equal and opposite the moment of the corresponding downward force on the bearing.

To use the apparatus, the beam is balanced on the knife-edges. The cylinder is fastened to the frame so that it can not rotate and a load W' is hung on the cord. If the knife-edges are in the correct position, the balance is unchanged when this load is applied. The cylinder is now released and is accelerated by the load. The part of the weight W' which accelerates the cylinder is transmitted to the axis of the bearing. This additional force downward on the left arm of the frame lifts the right arm. The poise p must now be moved toward the right to secure a running balance. If the acceleration of the cylinder is measured and if the moment of inertia of the cylinder is known, the balance reading makes it possible to determine the value of g. If g is known, the reading may be used to determine the moment of inertia of the cylinder. The results obtained by this apparatus require no correction for friction.

Example

A cylindrical drum with its axle weighs 8 pounds. Its diameter is a little less than 6 inches. A cord is wound several times round the drum and fastened to it. The radius of the drum plus the radius of the cord is exactly 3 inches. The drum is mounted on a frame similar to that of Fig. 301. The frame is supported on knife-edges which are on a line 3 inches from the axis of the axle. The poise p weighs 0.2 pound. When a mass of 1 pound is hung on the cord and the drum is released, the poise must be moved 0.96 foot toward the right to secure equilibrium. The pound mass moves 28.89 feet in 3 seconds after starting from rest. The value of g is known to be 32.16 feet per sec. per sec. at that locality. Find the moment of inertia of the drum.

The acceleration of the cord is

$$a = \frac{2 \times 28.89}{9} = 6.42 \text{ ft. per sec. per sec.}$$

$$\alpha = 25.68 \text{ radians per sec. per sec.}$$

$$T = 0.2 \times 0.96 = 0.192 \text{ ft. lb.}$$

$$I = \frac{Tg}{\alpha} = 0.2404$$

The force required to give an acceleration of 6.42 feet per second per second to the 1-pound mass is

$$P = \frac{1 \times 6.42}{32.16} = 0.1996 \text{ lb.}$$

The force required to produce a torque of 0.192 foot-pound on an arm 3 inches in length is $0.192 \div \frac{1}{4} = 0.768$ lb.

$$0.1996 + 0.768 = 0.9678 \text{ lb.}$$

The residue of 0.0322 lb. is taken up by the friction.

219. The Reactions of a System.—Figure 302 shows a frame which carries two pulleys. The frame is supported by two knife-edges. The line of these knife-edges passes through the center line of the cord which drives the pulleys. The cord is passed over pulley A and is wound round pulley B and fastened to it. If the frame is in equilibrium with no load on the cord, it is in equilibrium when the cord is loaded and pulley B is held by a brake or stop attached to the frame. The frame and the two pulleys may be regarded as a single body in equilibrium. The moment arm of a force applied by the cord is zero. The load on the cord, therefore, makes no change in the balance.

The frame and pulley B may be regarded as a rigid body, and pulley A may be treated separately. The simplest form of this apparatus is the one where the pulleys are of equal radii r.

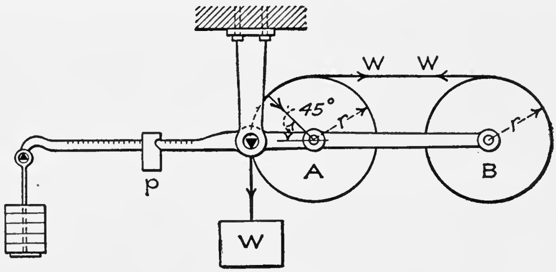

Fig. 302.

If pulley A is frictionless, the pull in the horizontal cord is equal to the weight W. The moment of this pull on pulley B about the knife-edges is equal to Wr. This moment is counter-clockwise. The resultant of the vertical pull W and the horizontal pull W acting on pulley A is a force of $2W \sin 45°$ through the axis of the pulley. The vertical component of this force is a force of $2W \sin 45° \cos 45°$ or W pounds. The moment of this

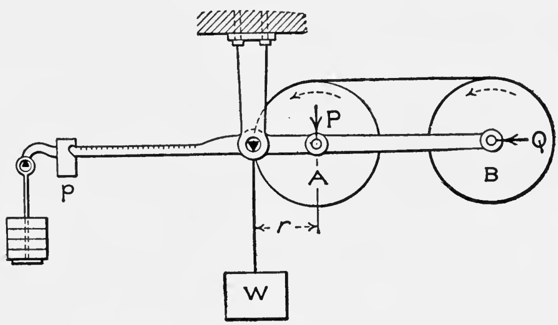

Fig. 303.

vertical component about the line of the knife-edges is Wr foot-pounds clockwise. The horizontal component has zero moment arm. The total moment is $Wr - Wr = 0$.

The pulley B may now be released and allowed to accelerate. The friction on either pulley has no effect on the equilibrium of the frame, for, in so far as friction is concerned, the pulleys and the frame may be regarded as a single rigid system.

If P is the force required to accelerate pulley A, there is a vertical force of P pounds downward from the pulley to the bearing. This force produces a clockwise moment Pr about the knife-edges, Fig. 303. If Q is the force which accelerates pulley B, a horizontal force equal to Q may be regarded as transferred from the top of pulley B to its axis, and the counter-clockwise

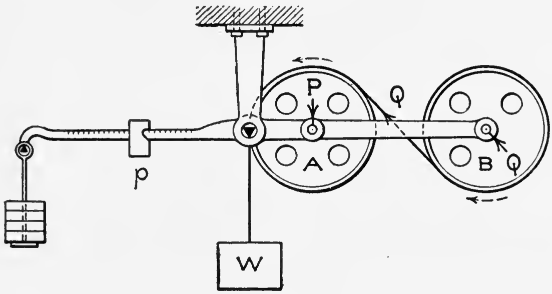

Fig. 304.

moment of the system is reduced by an amount Qr. The total counter-clockwise moment is reduced by an amount $(P + Q)r$ when the pulleys are released and allowed to accelerate. If the two pulleys have equal moments of inertia, the force P is equal to the force Q and the total change in moment is $2Pr$. To secure equilibrium, the poise p must be moved toward the left.

In Fig. 304, the cord passes around pulley B in the counter-clockwise direction. When this pulley is accelerated, the force Q which produces the acceleration may be regarded as shifted from the tangent to the axis. The change of moment is Qr. The clockwise moment on the system is reduced by the amount Qr. When the moment of inertia is the same for both pulleys, the counter-clockwise change in moment which is due to the acceleration of pulley B exactly balances the clockwise change which is due to the acceleration of pulley A. There is, therefore, no change of the balance of the frame, when the pulleys are of equal radius and equal moment of inertia, and rotate in opposite directions.

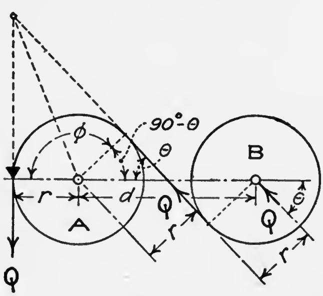

Fig. 305.

The discussion of the preceding paragraph may be stated in another way. In Fig. 305, the pulley A may be considered separately. The distance between the axes of the pulleys is d and the angle between the cord which joins the pulleys and the line which joins the axes is θ. When pulley B is accelerated by an effective force **Q**, the counter-clockwise moment about the knife-edge is $Q \sin \theta(d + r)$. Since $\sin \theta = \frac{2r}{d}$,

$$M = \frac{2Qr}{d}(d + r) = 2Qr + \frac{2Qr^2}{d}. \tag{1}$$

The resultant of two forces **Q** acting tangent to pulley A at an angle ϕ with each other is

$$R = 2Q \cos \frac{\phi}{2}. \tag{2}$$

This resultant passes through the axis of pulley A. Its vertical component is

$$V = 2Q \cos^2 \frac{\phi}{2} = Q(1 + \cos \phi) = Q(1 + \sin \theta) = Q\left(1 + \frac{2r}{d}\right) \tag{3}$$

The moment of this vertical force with respect to the knife-edge is

$$M = Qr + \frac{2Qr^2}{d}. \tag{4}$$

The total moment is the sum of the counter-clockwise moment of Equation (1) and the clockwise moment of Equation (4).

$$M_q = Qr \tag{5}$$

in the counter-clockwise direction. Equation (5) states that the change of moment on the frame when pulley B is accelerated is equal to the moment of the force which causes the acceleration.

Figure 306 shows the author's modification of the Atwood machine. A frame supported by knife-edges carries two equal pulleys. A cord which carries a mass m runs over pulley B and under pulley A. From pulley A the cord runs upward to a third pulley C which is carried on a separate support. The center line of this vertical cord intersects the line of the knife-edges which carry the frame. A mass W, which is larger than m is hung on the part of the cord which is suspended from C. (In Fig. 306 a fourth pulley D is shown, for clearness. A single pulley C is sufficient if placed with its axis parallel to the frame.

A brake (not shown in the drawing) holds pulley C stationary while the poise is adjusted to balance. When the brake is released, the mass m is accelerated upward. The additional force on the cord which supports the mass m is

$$P = \frac{ma}{g}. \tag{6}$$

This additional force is balanced by moving the poise p toward the left. A few trials are required to find the position of running equilibrium.

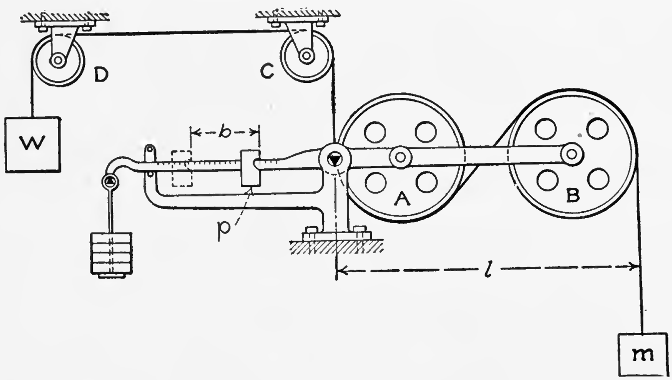

FIG. 306.

If l is the moment arm of the mass m, p is the mass of the poise, and b is the distance the poise is moved to secure equilibrium of the frame,

$$pb = \frac{ma}{g} l. \tag{7}$$

This apparatus is free from any correction for friction or the mass of the pulleys. To avoid any correction for the weight of the cord, it should be continuous from W to m. The mass m includes the mass of cord above and below it.

220. Summary.—Angular acceleration bears the same relation to linear acceleration that angular velocity bears to linear velocity.

$$\text{Angular acceleration} = \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

Angular acceleration is proportional to the torque and inversely proportional to the moment of inertia of the body.

$$\text{Torque} = \frac{\text{Moment of inertia} \times \text{angular acceleration}}{g};$$

$$T = \frac{I\alpha}{g}. \qquad \text{Formula XXX}$$

When the foot is the unit of length in the computation of torque and moment of inertia, the value of g is 32.174.

The component of the reaction at the axis in the direction normal to the plane through the axis and the center of mass is

$$N = P \cos \theta - \frac{ma}{g},$$

in which $P \cos \theta$ is the component of the applied force normal to the plane through the axis and the center of mass, m is the mass, and a is the linear acceleration of the center of mass. The component N is positive in the direction opposite the direction of the linear acceleration of the center of mass. The linear acceleration a is given by the equation

$$a = \bar{r}\alpha,$$

in which $\bar{r}$ is the distance of the center of mass from the axis of rotation.

The component of the reaction along the line, perpendicular to the axis, which joins the center of mass with the axis of rotation is equal to the component of the applied forces in this direction combined with the centrifugal force. If the axis of rotation passes through the center of mass, the centrifugal force is zero and the linear acceleration of the center of mass is zero; the reaction of the axis is equal and opposite to the applied force.

In the calculation of the acceleration of a rotating body, the actual mass may be replaced by an *equivalent mass* in the line of action of the applied force. If r' is the perpendicular distance from the axis to the line of action of the applied force, I is the moment of inertia of the body, and m' is the equivalent mass,

$$I = m'r'^2;$$

$$m' = \frac{I}{r'^2}.$$

By means of the equivalent mass, a problem of angular acceleration may be reduced to a problem of linear acceleration. The angular velocity and acceleration may finally be computed from the linear acceleration and velocity by dividing by r'.

CHAPTER XXIV

ANGULAR VIBRATION

221. Work of Torque.—When a force is applied to an arm of length r, the work done by the force when the arm turns through an angle θ is

$$U = Pr\theta, \qquad (1)$$

in which P is the component of the force normal to the plane through its point of application and the axis of rotation. Since the product $P \times r$ is the torque, Equation (1) is equivalent to

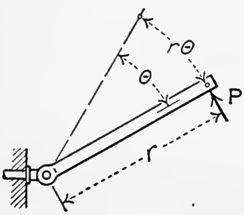

Fig. 307.

$$U = T\theta. \qquad (2)$$

Equation (2) states that the work of a constant force is equal to the product of the torque multiplied by the angular displacement. The work per revolution is

$$U = 2\pi T. \qquad (3)$$

When the torque varies,

$$U = \int T d\theta \qquad (4)$$

When an elastic rod, or a spiral or helical spring is twisted, the torque is proportional to the angular displacement. If K is the torque when the angular twist is one radian, the expression for the torque when the angle is θ radians is

$$T = K\theta. \qquad (5)$$

$$U = K\int \theta\, d\theta = \frac{K}{2}\Big[\theta^2\Big]_{\theta_1}^{\theta_2} = \frac{K}{2}(\theta_2^2 - \theta_1^2). \qquad (6)$$

When the initial angular displacement is zero

$$U = \frac{K\theta^2}{2}. \qquad (7)$$

Problems

1. A force of 20 pounds at the end of an arm 4 feet in length twists a steel rod through an angle of 90 degrees. If there is no initial torque, find the work. Solve by means of the average torque.

Ans. $U = \frac{80 + 0}{2} \times \frac{\pi}{2} = 62.832$ ft. lb.

2. A wheel is driven by a constant torque. Find the work per revolution.

Ans. $U = 2\pi T$.

3. A drum 16 inches in diameter is driven by a rope which is wound round it. The rope is $\frac{1}{2}$ inch in diameter and exerts a pull of 300 pounds. Find the work per revolution.

Ans. $U = 1295.9$ ft. lb.

222. Angular Velocity from Variable Torque.—When the torque is constant, the angular velocity is easily obtained first by finding the angular acceleration and then by multiplying by the time. When the torque is variable, the angular velocity at any given *position* is *easiest* found by means of the kinetic energy of rotation.

The most important problem is that in which the torque varies as the angular displacement. If a body on an elastic rod or spring is twisted through an angle β from the position of equilibrium, the work done is $\frac{K\beta^2}{2}$. This work is stored up as elastic energy. If the body is released and permitted to rotate back to the position at which its displacement from the position of equilibrium is θ, the change in elastic energy is the first member of the equation

$$\frac{K\beta^2}{2} - \frac{K\theta^2}{2} = \frac{I\omega^2}{2g}. \qquad \text{Formula XXXI}$$

The second member of Formula XXXI is the kinetic energy of rotation. The formula states that the change of potential energy is equal to the kinetic energy, or that the sum of the potential energy and the kinetic energy is constant. The angle β is the *amplitude* of the vibration in angular measure.

Formula XXXI reduces to

$$\omega^2 = \frac{Kg(\beta^2 - \theta^2)}{I}. \qquad (1)$$

The maximum angular velocity is

$$\omega^2 = \frac{Kg\beta^2}{I}. \qquad (2)$$

Problems

1. A homogeneous solid cylinder 1 foot in diameter and weighing 20 pounds is suspended with its axis vertical by means of a $\frac{1}{4}$-inch steel rod 10

24

feet in length, Fig. 308. A force of 3 pounds at the end of an arm 1 foot long is capable of twisting this rod through one radian. The cylinder is

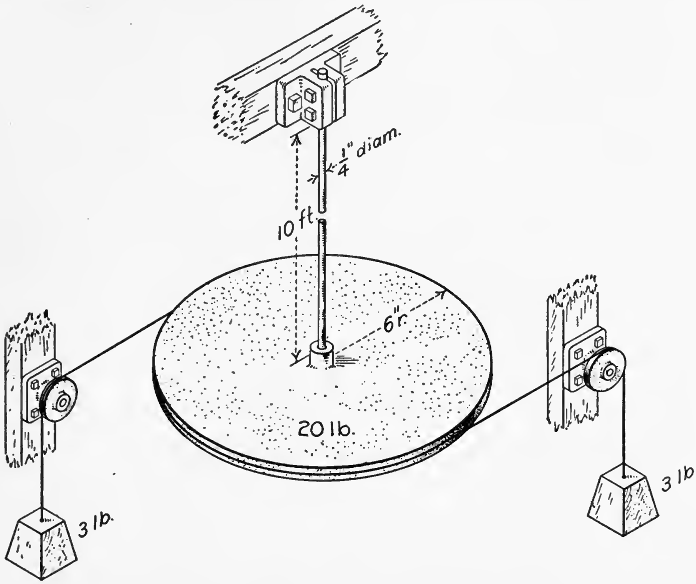

FIG. 308.

rotated through an angle of 180 degrees and then released. Find its maximum angular velocity.

Ans. $\omega^2 = \dfrac{3g\pi^2}{2.5}$; ω = 19.52 radians per sec.

2. In problem 1, what is the maximum angular velocity if the initial displacement is 90 degrees from the position of equilibrium?

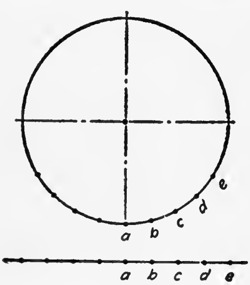

FIG. 309.

223. Time of Vibration.—Formula XXXI for the angular velocity is similar to Formula XXVI for the linear velocity caused by a force which varies as the displacement. In Formula XXXI, angular velocity and angular displacement take the place of linear velocity and displacement, and moment of inertia replaces mass.

If the circular motion of a point on a body which moves with the angular velocity of Formula XXXI is developed into linear motion, as shown in Fig. 309, this developed motion is simple harmonic. If a sheet of paper is placed on a cylinder which has this motion, and a pencil is moved with uniform speed parallel to its length, the pencil mark is the sine curve shown in Fig. 310.

Since the angular motion of a body subjected to torque which varies as the angular displacement is a simple harmonic motion, it may be studied by means of a circle of reference. The radius of this circle of reference is the angular amplitude β expressed in radians, Fig. 311. A point may be considered as moving in the circumference of this circle with a linear velocity which is equal to the maximum angular velocity of the vibrating mass. The time of a complete period

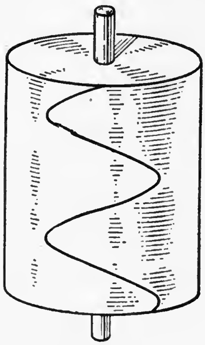

FIG. 310.

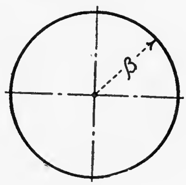

FIG. 311.

is the time required for this point to make one complete revolution around the circle of reference. The angular velocity at any instant is the component along the diameter of the circle of this maximum velocity in its circumference.

Problems

1. Find the time of a complete oscillation of the cylinder of Problem 1 of Art. 222.

$$\textit{Ans.} \quad t_c = \frac{2\pi^2}{\omega} = 2\pi\sqrt{\frac{2.5}{3g}} = 1.01 \text{ sec.}$$

2. Find the time of a complete oscillation for Problem 2 of Art. 222.

3. Express in terms of I and K the time of a complete oscillation caused by a torque which varies as the displacement. Compare the result with Equation (10) of Art. 185.

$$\textit{Ans.} \quad t_c = 2\pi\sqrt{\frac{I}{Kg}}. \qquad \text{Formula XXXII}$$

4. A cylindrical disk 6 inches in diameter is suspended with its axis vertical by means of a vertical wire. The disk makes 100 complete oscillations in 96 seconds. Two wires, each 0.04 inch in diameter, are fastened to the cylinder and extend horizontally to smooth pulleys, as shown in Fig. 308.

When a mass of 1 pound is hung on each wire, the cylinder is found to rotate 12 degrees. Find the moment of inertia of the cylinder.

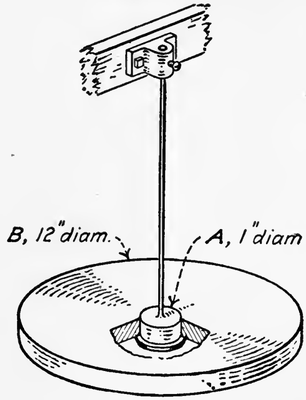

FIG. 312.

$$T = \frac{2 \times 3.02}{12}; \quad K = \frac{3.02}{6} \div \frac{\pi}{15} = \frac{15.10}{2\pi}.$$

$$t_c = 2\pi\sqrt{\frac{I}{Kg}}; \quad I = \frac{t_c^2 Kg}{4\pi^2}.$$

Log I = 0.25649, I = 1.805 in pound and foot units.

5. The small cylinder A of Fig. 312 suspended by a wire is found to make 1000 complete torsional vibrations in 240 seconds. A bronze disk B, 12 inches outside diameter, 1 inch inside diameter, and weighing 32.04 pounds is hung on the small cylinder. The time of 1000 complete vibrations is now 2500 seconds. Find the moment of inertia of the small cylinder.

Ans. I of disk $= 16.02 \times \frac{145}{576} = 4.0329; \quad \frac{I_0}{I_0 + I} = \frac{0.24^2}{2.5^2}$

$$\frac{I_0}{I} = \frac{0.0576}{6.25 - 0.0576}; \text{ Log } I_0 = \bar{2}.57417; \; I_0 = 0.0375.$$

6. Two wires, each 0.06 inch in diameter are attached to the disk of Problem 5 and passed over smooth pulleys. When a 1-pound mass is hung on each wire, the disk is twisted through an angle of 45° 50′. Find g.

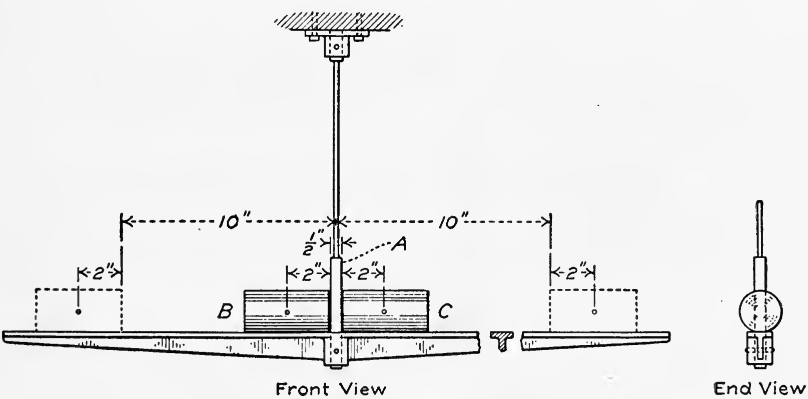

FIG. 313.

7. The frame of Fig. 313 is supported by a vertical wire at the center of the vertical cylinder A. Each of the cylinders B and C weighs 3 pounds and has its center of gravity 2 inches from the end nearest to cylinder A. When the inner ends of B and C are 0.250 inch from the vertical axis, the time of vibration of the system is 0.428 second. When cylinders B and C

are moved out till their ends are 10 inches from the vertical axis, the time of vibration is 1.126 second. Find the effective moment of inertia of the system with the cylinders in the first position.

Ans. $I_2 - I_1 = 5.7891$; $I_2 = 6.7667$; $I_1 = 0.9776$,

in which I_2 is the moment of inertia at the second position and I_1 is the moment of inertia at the first position.

8. A torque of 3.436 foot pounds twists the system of Problem 7 through an angle of 30 degrees. Find g.

The method of Problems 5 and 6 may be used to determine the value of g at any locality. The mass of the cylinder may be determined by accurate weighing. It is not necessary that standard weights be used in this weighing. It is necessary, however, that the weights used in weighing the disk and those used in determining the torque be expressed in terms of the same unit. This method assumes that the material of the disk is homogeneous throughout.

The method of Problems 7 and 8 does not require that the cylinders B and C should be homogeneous. The method, however, does require that the location of their centers of gravity be experimentally determined.

224. The Gravity Pendulum.—A body which is free to rotate through a small angle about a horizontal axis under the action of its weight is called a pendulum. The axis O of Fig. 314 about which it rotates is called the axis of suspension. If m is the mass of the pendulum and $\bar{r}$ is the distance of its center of mass below the axis of suspension, the torque which is due to its weight, when the pendulum is displaced through an angle θ from the position of equilibrium is given by the equation

$$T = m\bar{r}\sin\theta \qquad (1)$$

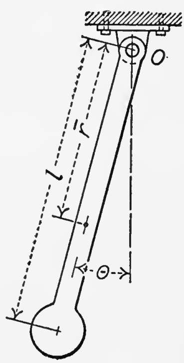

FIG. 314.

If θ is small, its sine is practically equal to its arc in radian measure. For a small angle, Equation (1) becomes

$$T = m\bar{r}\theta. \qquad (2)$$

When the angular displacement is small, the torque is proportional to the displacement and Formula XXXII applies to the motion of the pendulum. The value of K for a pendulum vibrating through a small arc is $K = m\bar{r}$, and Formula XXXII becomes,

$$t_c = 2\pi\sqrt{\frac{I}{m\bar{r}g}} \qquad (3)$$

Problems

1. A uniform bar of length l vibrates as a pendulum about an axis perpendicular to its length through one end. Find the time of a complete period.

Ans. $t_c = 2\pi\sqrt{\frac{2l}{3g}}$.

2. Solve Problem 1 for a length of 4 feet.

Ans. $t_c = 1.809$ sec.

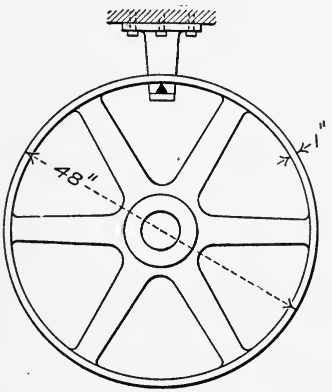

Fig. 315.

3. A flywheel is 4 feet in diameter, weighs 200 pounds, and has its center of mass at its axis. The flywheel vibrates as a pendulum about a knife edge under the rim at a distance of 23 inches from the axis (Fig. 315). The wheel makes 100 complete vibrations in 196 seconds. Find its moment of inertia with respect to the knife edge and its moment of inertia with respect to its axis.

Ans. $I = 1200.2$; $I_0 = 465.5$.

225. Simple Pendulum.—The ideal simple pendulum consists of a small body suspended by a weightless cord. The dimensions of the body must be so small that all parts are at the same distance from the axis of suspension. If m is the mass of the body and l is its distance from the axis of suspension,

$$\bar{r} = l\,;\, I = ml^2,$$

and Equation (3) of the preceding article becomes

$$t_c = 2\pi\sqrt{\frac{l}{g}}. \tag{1}$$

Problems

1. Find the length of the simple pendulum which makes a single oscillation in one second at a place where $g = 32.174$ ft.

Ans. $l = 39.12$ in.

2. Find the value of g at a place where the "second pendulum" is 39.00 inches in length.

In the ordinary physical pendulum,

$$I = mk^2 = m(\bar{r}^2 + k_0^2), \tag{2}$$

in which k_0 is the radius of gyration with respect to the axis parallel to the axis of suspension which passes through the center of mass. The expression for the time of a complete period is

$$t_c = 2\pi \sqrt{\frac{\bar{r} + \frac{k_0^2}{\bar{r}}}{g}}. \qquad (3)$$

The term $\bar{r} + \frac{k_0^2}{\bar{r}}$ in Equation (3) is a length and corresponds with l of Equation (1). This term is called the *length* of the *equivalent simple pendulum.* It will be represented by l'.

$$l' = \bar{r} + \frac{k_0^2}{\bar{r}}. \qquad (4)$$

If k_o is small compared with $\bar{r}$, then l' approaches $\bar{r}$, and the pendulum is approximately a simple pendulum.

Problem

3. A sphere 0.4 inch in diameter is suspended by a cord of negligible weight. The center of the sphere is 40 inches from the point of suspension of the cord. What is the length of the equivalent simple pendulum? What is the relative error in the time of vibration if the radius of gyration of the sphere is neglected?

Ans. $l' = 40.004$ in.; error 1 part in 20,000.

226. Axis of Oscillation.—The *axis* of *oscillation* of a physical pendulum is a line parallel to the axis of suspension at a distance from the axis of suspension equal to the length of the equivalent simple pendulum. The axis of oscillation lies in the plane which passes through the axis of suspension and the center of mass of the pendulum. A particle at the axis of oscillation is neither accelerated nor retarded by the motion of the pendulum as a whole. A particle below the axis of oscillation is forced to vibrate faster than it would if it were suspended by a weightless cord and moved as a simple pendulum. A particle above the axis of oscillation moves slower than it would if it were moving alone.

If l' is the distance between the axis of suspension and the axis of oscillation,

$$l' = \frac{k^2}{\bar{r}} = \bar{r} + \frac{k_0^2}{\bar{r}}.$$

Problems

1. A bar of uniform section and density vibrates about an axis which is perpendicular to its length and passes through one end. Find the distance between the axis of suspension and the axis of oscillation.

Ans. $l' = \frac{2l}{3}.$

2. A uniform bar of length l vibrates as a pendulum about an axis perpendicular to its length at a distance of one-third its length from one end. Find the length of the equivalent simple pendulum.

Ans. $l' = \frac{2l}{3}$.

3. Find the length of the simple pendulum equivalent to a uniform bar of length l which vibrates about an axis of suspension at one-fourth the length from one end. Solve also when the axis of suspension is three-eighths the length from one end.

Ans. $l' = \frac{7l}{12}$; $l' = \frac{19l}{24}$.

4. What is the position of the axis of suspension of a uniform bar which gives the minimum time of vibration? What is the length of the equivalent simple pendulum?

Ans. $\bar{r} = \frac{l}{\sqrt{12}}$; $l' = \frac{l}{\sqrt{3}} = 0.577\ l$.

227. Exchange of Axes.—If the axis of oscillation of a pendulum is made the axis of suspension, the former axis of suspension becomes the axis of oscillation. When the original axis of suspension is at a distance $\bar{r}$ from the center of gravity of the pendulum, the length of the equivalent simple pendulum is

$$l' = \bar{r} + \frac{k_0^2}{\bar{r}}. \tag{1}$$

The distance from the center of gravity to the axis of oscillation is $\frac{k_0^2}{\bar{r}}$. When the axis of oscillation becomes the axis of suspension, $\frac{k_0^2}{\bar{r}}$ is the distance from the new axis of suspension to the center of gravity. If l'' is the length of the equivalent simple pendulum when the axis of oscillation is made the axis of suspension,

$$l'' = \frac{k_0^2}{\bar{r}} + \frac{k_0^2}{\frac{k_0^2}{\bar{r}}} = \frac{k_0^2}{\bar{r}} + \bar{r}, \tag{2}$$

$$l'' = l'. \tag{3}$$

The principle of Equation (3) is employed in the determination of g. A pendulum is provided with a pair of knife-edges parallel to each other at approximately the correct distance for one knife-edge to be the axis of suspension and the other the axis of oscillation. A small body, b of Fig. 316, is arranged to slide along the pendulum. This body is adjusted till the time of vibration is the same when either axis is used as the axis of suspension. The time of vibration and the distance between the knife-edges afford the necessary data for the determination of g by substitution in the equation of vibration of the simple pendulum.

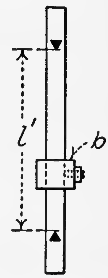

FIG. 316.

Instead of one adjustment weight, two equal weights may be employed. These may be moved equal distances in opposite directions at each adjustment. When the adjustment is made in this way, $\bar{r}$ is not changed, the variation of time of vibration follows a simpler law, and correct position is easier to find.

The determination of g by this method involves only measurements of time and length, and is independent of mass and density.

228. Summary.—When the torque varies as the angular displacement, the work is given by

$$U = \frac{K}{2}(\theta_2^2 - \theta_1^2),$$

in which K is the torque when the displacement from the position of zero torque is one radian, and θ_1 and θ_2 are angular displacements from the position of equilibrium. When the initial displacement is zero and the final displacement is θ radians,

$$U = \frac{K\theta^2}{2}.$$

The equation which connects the work of torque and the kinetic energy is

$$\frac{K\beta^2}{2} - \frac{K\theta^2}{2} = \frac{I\omega^2}{2g}, \qquad \text{Formula XXXI}$$

in which β is the angular displacement when the velocity is zero, and ω is the angular velocity when the displacement is θ. When θ is zero

$$\omega^2 = \frac{Kg\beta^2}{I}.$$

If angular displacement is plotted on a straight line, the motion of a body subjected to torque which varies as the displacement is a simple harmonic motion. The time of a single oscillation is

$$t = \pi\sqrt{\frac{I}{Kg}}.$$

The time of a complete period is

$$t_c = 2\pi\sqrt{\frac{I}{Kg}}. \qquad \text{Formula XXXII}$$

In a gravity pendulum the torque is

$$T = m\bar{r}\sin\theta.$$

When the angle is small, $\sin\theta$ reduces to θ and

$$T = m\bar{r}\theta.$$

With a small amplitude of vibration the torque on a gravity pendulum varies as the displacement and Formula XXXI applies. The constant K is equal to $m\bar{r}$, and the time of a complete period is

$$t_c = 2\pi\sqrt{\frac{I}{m\bar{r}g}} = 2\pi\sqrt{\frac{\bar{r} + \frac{k_0^2}{\bar{r}}}{g}}.$$

The term $\bar{r} + \frac{k_0^2}{\bar{r}}$ is the length of the equivalent simple pendulum. The length of the equivalent simple pendulum is the distance between the axis of suspension and the axis of oscillation.

The axis of suspension and the axis of oscillation are interchangeable. If the time of vibration is found for a given axis of suspension and the pendulum is then reversed and suspended from the axis of oscillation, the time of vibration for the two positions is found to be identical.

CHAPTER XXV

MOMENTUM AND IMPULSE

229. Momentum.—The product of the mass of a body multiplied by its velocity is called its *momentum.*

$$\text{Momentum} = mv. \tag{1}$$

Since velocity is a vector quantity and mass is a scalar quantity, their product is a vector. Momentum has direction as well as magnitude.

Problems

1. What is the total momentum of 40 pounds moving east 50 feet per second and 20 pounds moving west 60 feet per second?

Ans. Momentum = 800 in foot and pound units.

2. What is the total momentum of 20 pounds moving north 30 feet per second and 40 pounds moving east 20 feet per second?

Ans. Momentum = 1000 units north 53° 08′ east.

230. Impulse.—The rate of change of momentum when the velocity changes is $m\frac{dv}{dt}$, which is equal to ma.

$$\text{Force} = \frac{ma}{g} = \frac{m\frac{dv}{dt}}{g}, \tag{1}$$

$$Pdt = \frac{mdv}{g}; \tag{2}$$

$$\int Pdt = \frac{m}{g}\int dv; \tag{3}$$

$$\int Pdt = \frac{m}{g}(v_2 - v_1). \tag{4}$$

The term $\int Pdt$ is called the impulse of the force. When the force is constant during an interval of time t, the impulse is Pt. Equation (4) shows that the change of momentum is proportional to the impulse. When the pound force is used as the unit of force and the pound mass is used as the unit of mass, the impulse is the change in momentum divided by 32.174. In terms of the absolute units, the change of momentum is numerically equal to the impulse.

When an impulse is given by a blow, as when an object is struck by a hammer, the force is large and the time is very short. Such an impulse is frequently called an *instantaneous impulse.* *Sudden impulse* is a better name. Suppose that a ball moving with a velocity of 100 feet per second strikes a wall and bounds back with a velocity of nearly 100 feet per second. Suppose that the compression of the ball amounts to 0.5 inch. If the force were uniform during the time of contact, the average velocity would be 50 feet per second while the ball is coming to rest. The time required to bring the ball to rest would be

$$\tfrac{1}{24} \div 50 = \tfrac{1}{1200} \text{ second.}$$

The entire time of contact would be about $\frac{1}{600}$ second. If the pressure varies as the displacement, instead of being uniform, the motion during contact is simple harmonic. The time required to complete a semicircle of $\frac{1}{2}$ inch radius at a speed of 100 feet per second is $\dfrac{\pi}{2400} = 0.0013$ second. When a hammer strikes an anvil, the deformation is much smaller and the time of contact is less. In general, the duration of a so-called instantaneous impulse is a few ten-thousandths of a second.

It is sometimes stated that an instantaneous impulse is completed before the body which is struck starts to move. This statement is incorrect. The full velocity is attained at the end of the impulse. The time, however, is so short that the actual displacement is small. In the case of a galvanometer needle, which is actuated by the discharge of a condenser, the needle has attained its maximum velocity at the time when the current becomes negligible. The deflection, on the other hand, is so small that the moment arm of the impulse is practically constant throughout the entire impulse.

231. Action and Reaction.—When force acts on a body, the force must come from some other body. Since, as stated by Newton's Third Law, action and reaction are equal, the force exerted by the second body on the first body is equal and opposite the force exerted by the first body on the second body. Since the forces are equal, the impulses are equal, and, consequently, the change of momentum of one body is equal and opposite the change of momentum of the other.

In Fig. 317, *A* and *B* are two bodies. Suppose that *A* exerts a force on *B* by means of a spring which forms a part of *A*. The change of momentum of A toward the left is equal and opposite the change of momentum of B toward the right. Suppose that A with the attached spring weighs 5 pounds and B weighs 4 pounds, and suppose that both are stationary. When the spring is released, suppose that B receives a velocity of 10 feet per second toward the right. Its change of momentum is 40. At the same time, A receives a velocity of 8 feet per second toward the left. Its change of momentum is $^{-}40$. The total change of momentum of the two bodies considered as a single system is zero. It is impossible to change the total momentum of two bodies by means of any force exerted between them.

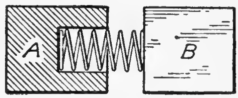

FIG. 317.

Any change in the momentum of a body is at the expense of an equal and opposite change in the momentum of some other body or bodies. When a man jumps upward into the air, he kicks the earth in the opposite direction with equal momentum. While he is in the air, the attraction of gravity draws him toward the earth and, likewise, draws the earth toward him. Considered as one body, the center of gravity of the earth and the man moves on unchanged.

If m_1 is the mass of one body and v_1 is its velocity, m_2 is the mass of a second body and v_2 is its velocity, and if a force acts between the bodies which changes the velocity of m_1 from v_1 to v_1' and changes the velocity of m_2 from v_2 to v_2',

$$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'. \qquad (1)$$

Example

A mass of 20 pounds moving east 30 feet per second overtakes a mass of 12 pounds moving east 12 feet per second. After collision, the bodies move together with the same velocity. Find that velocity.

The total momentum east is

$$20 \times 30 = 600$$
$$12 \times 10 = 120$$
$$\overline{720}$$

The total mass is 32 pounds. If v is the final velocity,

$$32v = 720$$
$$v = 22.5 \text{ ft. per sec. east.}$$

Problems

1. A man weighing 150 pounds jumps horizontally with a speed of 10 feet per second from a boat weighing 200 pounds. If the boat was initially at rest and if the friction of the water is neglected, with what velocity does the boat move in the opposite direction?

Ans. 7.5 ft. per sec.

2. A man weighing 150 pounds can jump with a horizontal velocity of 15 feet per second with reference to a fixed point. With what speed relative to the earth can he jump from a boat which weighs 100 pounds and what is the velocity of the boat?

Ans. 6 ft. per sec.; 9 ft. per sec.

3. A shot weighing 10 pounds is fired from a gun weighing 800 pounds. The velocity of the shot relative to the earth is 1200 feet per second. Find the velocity of the gun's recoil. Find the kinetic energy of the shot and of the gun.

Ans. $v = 15$ ft. per sec.; kinetic energy of shot $= 223{,}780$ ft. lb., kinetic energy of gun $= 2797$ ft. lb.

4. When a shot is fired from a gun, show that the kinetic energy of the shot is to the kinetic energy of the gun as the weight of the gun is to the weight of the shot.

5. A 4-pound mass moving east 60 feet per second meets a 5-pound mass moving west 20 feet per second. After collision, the 5-pound mass moves east 20 feet per second. By means of the total momentum find the velocity of the 4-pound mass.

Ans. $v = 10$ ft. per sec. east.

6. If the 5-pound mass of Problem 5 moves east 40 feet per second after collision, what is the velocity of the 4-pound mass?

Ans. $v = 15$ ft. per sec. west.

7. The bodies of Problem 5 collide obliquely. After collision the 5-pound mass moves northeast with a velocity of 30 feet per second. What is the velocity of the 4-pound mass?

Ans. The 4-pound mass has a velocity component east of 10.97 ft. per sec. and a component south of 26.51 ft. per sec.

232. Collision of Inelastic Bodies.—When inelastic bodies collide, they remain together after collision and move with the same velocity. If m_1 and m_2 are the masses, v_1 and v_2 are their respective velocities before collision, and v is their common velocity after collision, the momentum equation is

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v. \qquad (1)$$

When inelastic bodies collide, there is a loss of kinetic energy. Some of the kinetic energy is transformed into heat energy.

Problems

1. A mass of 20 pounds moving east with a velocity of 40 feet per second overtakes a mass of 30 pounds moving east with a velocity of 10 feet per second. After collision the bodies move together. Find their common velocity. *Ans.* $v = 22$ ft. per sec.

2. In Problem 1, what part of the kinetic energy is lost?

Ans. 41 per cent.

3. A mass of 20 pounds moving east with a velocity of 40 feet per second meets a mass of 30 pounds moving west with a velocity of 10 feet per second. Find the velocity after collision and the kinetic energy lost.

Ans. 10 ft. per sec.; 87.8 per cent.

4. A gun is hung in a horizontal position on a vertical support which is hinged at the top as shown in Fig. 318. The gun and support weigh 800 pounds and their center of gravity is 12 feet below the hinge. When a 10-pound shot is fired from the gun, the reaction swings the support through an arc of 45 degrees. Find the velocity of the shot.

Ans. $v = 1203$ ft. per sec.

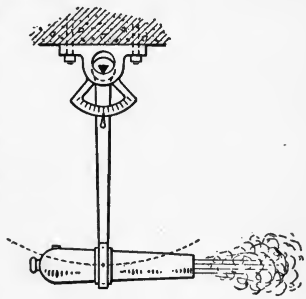

FIG. 318.

5. A soft wood block weighing 5 pounds is suspended by vertical cords 5 feet in length. A bullet weighing 0.02 pound is fired horizontally into the block and remains imbedded in it. The block swings through a vertical angle of 35 degrees. Find the velocity of the bullet.

233. Collision of Elastic Bodies.—When two elastic bodies collide, both bodies suffer a change of form. Their centers of mass continue to approach each other for a little time and both bodies are compressed at the surface of contact. If one of the bodies is fixed so as to be practically stationary, the other body will come to rest and then rebound. If both bodies are free to move, there comes a time when each is stationary relative to the other, and both are moving with the same velocity relative to the earth. Fig. 319 represents three stages of the collision of two bodies. In Fig. 319, I, the bodies are moving toward each other at the beginning of contact. In Fig. 319, II, the centers O_1 and O_2 are nearest to each other. The bodies are stationary relative to each other and are moving with a velocity v relative to the earth. It is assumed that the momentum of m_1 is the greater; the common velocity is, therefore, toward the right. The interval from the beginning of collision to the condition of Fig. 319,

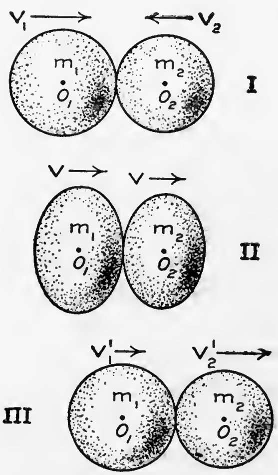

FIG. 319.

II, is called the *compression stage* of the collision. The common velocity v at the end of the compression stage is found in the same way as if the bodies were inelastic.

After the bodies have reached the end of the compression stage, their elasticity causes them to separate. The interval from the end of the compression stage to the time when the bodies cease to be in contact with each other is called the *restitution stage.* Fig. 319, III, shows the bodies at the end of the restitution stage.

If the bodies are perfectly elastic, the impulse during restitution is equal to the impulse during compression and the change of momentum of each body during restitution is equal to the change during compression. With imperfectly elastic bodies the change of momentum during restitution is less than the change during compression. The ratio of the impulse during restitution to the impulse during compression is called the *coefficient* of *restitution.* When the bodies are perfectly elastic, the coefficient of restitution is unity. The coefficient of restitution is expressed by the letter e.

Problems

1. A body is thrown against a fixed wall with a speed of 60 feet per second and rebounds with a speed of 45 feet per second. Find the coefficient of restitution. *Ans.* $e = 0.75$.

2. A ball is dropped 4 feet upon a horizontal surface and rebounds 3 feet. Find the coefficient of restitution. *Ans.* $e = 0.866$.

3. A body is dropped 10 feet to a horizontal surface. If the coefficient of restitution is 0.8, how high will it rebound?

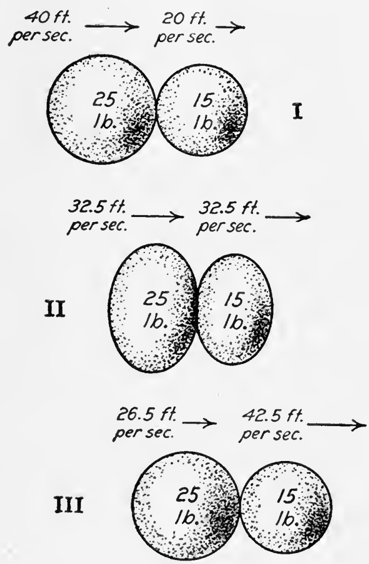

FIG. 320.

The behavior of a pair of elastic bodies during collision is best studied by a numerical example rather than by the use of literal formulas. Fig. 320, I, represents a mass of 25 pounds moving east with a velocity of 40 feet per second, which is overtaking a mass of 15 pounds moving east with a velocity of 20 feet per second. Fig. 320, II, shows the two bodies at the end of the compression stage. It may be assumed that the bodies are photographed and that the plate is shifted vertically between exposures. The initial momentum is

$$25 \times 40 + 15 \times 20 = 1300.$$

$$v = 32.5 \text{ feet per second.}$$

As a condition of the problem, it will be assumed that the coefficient of restitution is 0.8. The change of velocity of the 25-pound mass during compression is $32.5 - 40 = -7.5$ feet per second. The change of velocity of this mass during restitution is $0.8\,(-7.5) = -6$ feet per second. The final velocity of the 25-pound mass is

$$v = 32.5 - 6 = 26.5 \text{ feet per second east.}$$

The change of velocity of the 15-pound mass during compression is 12.5 feet per second. Its change during restitution is 10 feet per second. The final velocity of the 15-pound mass is

$$v = 32.5 + 10 = 42.5 \text{ feet per second east.}$$

The results may be checked by means of the final momentum;

$$\begin{aligned} 25 \times 26.5 &= 662.5 \\ 15 \times 42.5 &= 637.5 \\ \hline \text{Total momentum} &= 1300. \end{aligned}$$

Problems

1. A mass of 40 pounds moving east with a velocity of 30 feet per second meets a mass of 20 pounds moving west with a velocity of 45 feet per second. The coefficient of restitution is 0.6. What is the velocity of each body after collision? *Ans.* 10 ft. per sec. west; 35 ft. per sec. east.

2. A mass of 20 pounds moving east 40 feet per second meets a mass of 5 pounds moving west 50 feet per second. After collision, the 20-pound mass moves east 8.5 feet per second. By equating the momenta, find the velocity of the 5-pound mass after collision. *Ans.* $v = 76$ ft. per sec. east.

3. In Problem 2, what is the common velocity at the end of the compression stage and what is the coefficient of restitution?

Ans. $v = 22$ ft. per sec. east; $e = 0.75$.

4. A mass of 15 pounds moving east 60 feet per second overtakes a mass of 10 pounds moving east 20 feet per second. After collision, the mass of 10 pounds moves east 65.6 feet per second. Find the final velocity of the mass of 15 pounds and find the coefficient of restitution.

5. A baseball moving 100 feet per second is struck by a bat moving 60 feet per second in the opposite direction. If the bat weighs 6 times as much as the ball, and the coefficient of restitution is 0.9, with what velocity will the ball leave the bat? *Ans.* 133.6 ft. per sec.

6. In Problem 1, what is the energy of each mass before and after collision; what is the total loss of energy; and what is the ratio of the loss to the original energy? *Ans.* Loss = 621.6 ft. lb. = 62.7% of initial energy.

7. Solve Problem 1 if the bodies are perfectly elastic, and find the energy loss. *Ans.* 20 ft. per sec. west; 55 ft. per sec. east. No energy lost.

234. Moment of Momentum.—If a body of mass m is moving with linear velocity v, its momentum is mv. If O, Fig. 321, is an axis perpendicular to the plane of the paper, and if r is the length of the line perpendicular to this axis and to the line of motion of the mass m, the product of the momentum of m multiplied by the length r is the *moment of momentum* of m with respect to the axis. When a body is rotating about an axis, the linear velocity of an element dm at a distance r from the axis is given by the equation

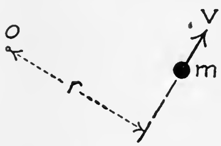

FIG. 321.

$$v = r\omega \tag{1}$$

The momentum of this element is $r\omega dm$ and the moment of momentum with respect to the axis of rotation is $r^2\omega dm$. The total moment of momentum is

$$\omega\int r^2 dm = \omega I. \tag{2}$$

The moment of momentum with respect to an axis of a body which is rotating about that axis is the product of the moment of inertia of the body multiplied by its angular velocity. A comparison of moment of momentum with momentum shows that moment of inertia replaces mass, and angular velocity replaces linear velocity.

To change the momentum of a body, force must be applied from an outside body. Likewise, to change the moment of momentum of a body, torque must be applied from some outside body or system of bodies. From Formula XXX

$$T = \frac{I\alpha}{g} = \frac{I}{g}\,\frac{d\omega}{dt}; \tag{3}$$

$$\int T dt = \frac{I}{g}\int d\omega; \tag{4}$$

$$Tt = \frac{I\omega}{g} + C, \tag{5}$$

in which C is an integration constant. The product Tt may be called the impulse of the torque.

The moment of momentum of a rotating body may be changed by changing the angular velocity or by changing the configuration of the body in such a way as to alter its moment of inertia. Fig. 322 shows a frame arranged to rotate about a vertical axis. The frame carries two equal bodies A and B. These bodies are connected to cords which run over pulleys and are attached to the sleeve F on the vertical shaft. Suppose that the frame is rotating with an angular velocity ω when A and B are in the positions shown in the figure. If the sleeve F is now pulled down by some outside force, or by its own weight, and draws A and B in toward the axis, the moment of inertia of the system is reduced. Since

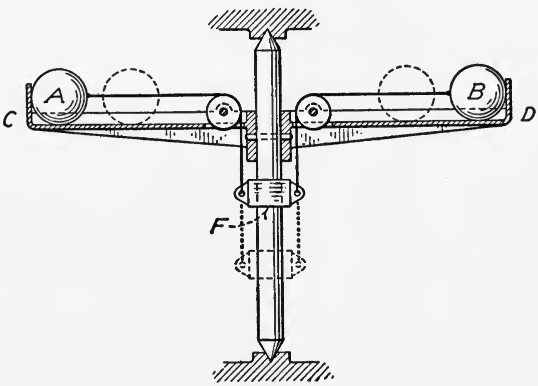

FIG. 322.

the moment of momentum remains constant, the angular velocity of the frame must increase. If the sleeve is lifted and the bodies A and B are allowed to move out to the original positions, the moment of inertia is again increased and the angular velocity is correspondingly diminished.

In Fig. 323, a mass m is supposed to be revolving in a horizontal plane about a vertical axis perpendicular to the plane of the paper at O. The mass is at a distance r from the axis and may be regarded as supported by a frictionless horizontal plane and held by a cord attached to the axis. If the cord is released at the axis, the mass will move along the line of the tangent to its path at the moment of release. If the angular velocity is ω_1, the linear velocity in the tangent line will be $r\omega_1$. Suppose that the cord is stopped at the axis when the mass is in the position of Fig. 323, II, at a distance R from the axis. The linear velocity of the mass is still $r\omega_1$. The component of this linear velocity perpendicular to the radius R is $r\omega_1 \cos \theta$. Its angular velocity about the axis is this component of the linear velocity divided by R

$$\omega = \frac{r\omega_1 \cos \theta}{R}. \tag{6}$$

Since $\cos \theta = \dfrac{r}{R}$,

$$\omega = \frac{r^2\omega_1}{R^2}; \tag{7}$$

$$R^2\omega = r^2\omega_1, \tag{8}$$

in which ω is the angular velocity at the new position. The moment of inertia at the first position is mr^2 and at the second

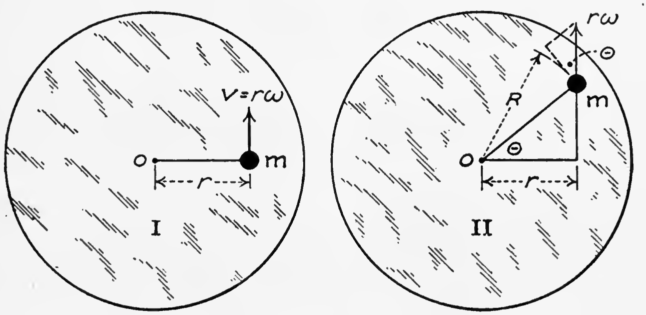

FIG. 323.

position is mR^2. Multiplying both sides of Equation (8) by m,

$$I\omega = I_1\omega_1, \tag{9}$$

in which I_1 is the moment of inertia at the first position at a distance r from the axis, and I is the moment of inertia at the second position at a distance R from the axis. It may be assumed that the dimensions are relatively so small that the moment of inertia of the body with respect to an axis through the center of mass is negligible. This assumption, however, is not necessary, since the body may be considered as composed of a great number of small elements, each of which has its own r and R.

Equation (9) gives an independent illustration of the fact that the moment of momentum is not altered when the configuration is changed by forces which exert no torque about the axis. On the other hand, the kinetic energy of the system is changed.

The kinetic energy varies as the square of the angular velocity while the moment of momentum varies as the first power of the angular velocity. When the moment of inertia is doubled, the angular velocity becomes one-half as great and the kinetic energy becomes one-half as great. In the system of Fig. 323, the kinetic energy is expended when the radial motion is stopped by the inelastic cord. If the body is brought back toward the axis, considerable force must be exerted on the cord. The work done on the system through the cord represents the increase of kinetic energy.

Example

A frame similar to Fig. 322 carries two masses, each of which weighs 3 pounds. The moment of inertia of the frame about its axis, together with the moment of inertia of each mass about an axis through its center of mass parallel to the axis of the frame is 2 units (expressed in pounds and feet). When the masses are each 1 foot from the vertical axis, the system is rotating with an angular velocity of 10 radians per second. The sleeve attached to the cords is then raised and the masses move to positions 2 feet from the axis. Find the angular velocity and the change of kinetic energy.

In the first position

$$I_1 = 2 + 6 = 8;$$

Moment of momentum $= 8 \times 10 = 80.$

In the second position

$$\begin{aligned} I_2 &= 2 + 6 \times 4 = 26; \\ 26\omega_2 &= 80; \\ \omega_2 &= 3.08 \text{ radians per sec.} \end{aligned}$$

In the first position, the kinetic energy is 12.43 foot-pounds. In the second position, the kinetic energy is 4.86 foot pounds.

Problems

1. In the above example the two masses are drawn in till their centers of gravity are each 6 inches from the axis. Find the angular velocity.

Ans. $\omega = 22.86$ radians per second.

2. A circular disk 2 feet in diameter and weighing 4 pounds rotates about a vertical axis with an angular velocity of 6 radians per second. A second disk 1 foot in diameter and weighing 16 pounds is coaxial with the first disk. This second disk is dropped a short distance upon the first disk and adheres to it. If the second disk is initially stationary, what is the common velocity of the two when they rotate together? *Ans.* $\omega = 3$ radians per second.

3. Solve Problem 2 if the axis of the second disk is 6 inches from the axis of rotation.

4. Solve Problem 2 if the second disk has an initial velocity of 10 radians per second in the same direction as the first disk.

5. Figure 324 represents a frame arranged to rotate about a vertical axis with little friction. A mass m rolls on a curved arm. The moment of inertia of the frame about its axis together with the moment of inertia of m about a vertical axis through its center of gravity is I. When the mass m is at a distance r_1 from the axis of rotation, the angular velocity of the frame is ω_1 radians per second. Find the form of the arm in order that the mass m may be in equilibrium in any position.

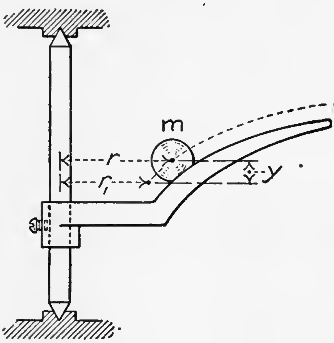

Fig. 324.

The centrifugal force is $\frac{mr\omega^2}{g}$, in which ω is the angular velocity when the center of mass is at a distance r from the axis.

When the center of mass is at a distance r_1 from the axis, the moment of inertia is $I + mr_1^2$; and when m is at a distance r from the axis, the moment of inertia is $I + mr^2$. If ω is the angular velocity when the center of mass of m is at a distance r from the axis, the equation of moment of momentum is

$$(I + mr_1)\omega_1 = (I + mr^2)\omega.$$

$$\omega^2 = \frac{(I + mr_1^2)^2\omega_1^2}{(I + mr^2)^2}.$$

By the method of virtual work, the equation of the surface of equilibrium is given by the expression

$$Hdx + Vdy = 0.$$

The centrifugal force is H and the weight of m is V. Since the weight is downward, $V = -m$. Substituting the value of ω^2 in the expression for the centrifugal force and replacing dx by dr, the equation of the surface becomes,

$$mdy = \frac{m(I + mr_1^2)^2\omega_1^2\, r\, dr}{g(I + mr^2)^2};$$

$$y = \frac{-(I + mr_1^2)^2\omega_1^2}{2gm(I + mr^2)} + C.$$

If $y = 0$ when $r = r_1$,

$$C = \frac{(I + mr_1^2)\omega_1^2}{2gm};$$

$$y = \frac{(I + mr_1^2)\omega_1^2}{2gm}\left(1 - \frac{I + mr_1^2}{I + mr^2}\right).$$

When the angular velocity exceeds ω_1 by ever so little, the mass m moves out to the extremity of the arm and the angular velocity falls. The kinetic energy of rotation is diminished, while the potential energy is increased. When the velocity is slightly diminished, the mass m moves inward toward the axis, and the angular velocity increases.

235. Center of Percussion.—The center of percussion of a body with respect to an axis is the point at which a sudden impulse normal to the plane through the axis and the center of mass may be given to the body without causing an additional reaction at the axis in the direction of the impulse. Fig. 325 represents a bar hinged at the top. If a blow perpendicular to its length is delivered to the bar at the middle, the bar tends to move toward the left parallel to itself. In order that it may rotate around the hinge, the hinge must exert a force toward the right. If the blow is delivered near the bottom, as shown in Fig. 325, II, the bar tends to rotate in a clockwise direction about some axis below the hinge. The hinge must then exert a reaction toward the left in the direction of the impulse. Between the middle of the bar and the bottom there is some point at which a blow may be struck without developing any horizontal reaction in the hinge. This point is the center of percussion. Students who play baseball are familiar with the effects of failure to strike a ball at the center of percussion of the bat.

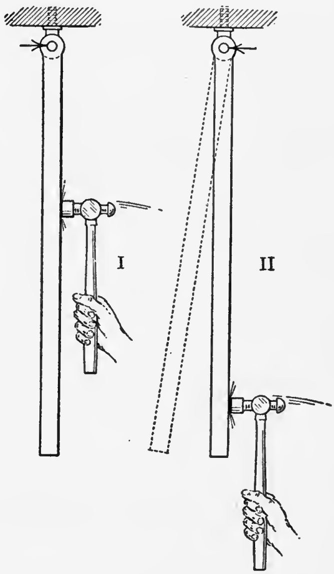

FIG. 325.

If P is the force on a body such as the bar of Fig. 325, a is the linear acceleration of the center of mass, and N is the normal component of the hinge reaction opposite the direction of P. Formula XXII gives the equation

$$P - N = \frac{ma}{g}. \tag{1}$$

If $N = 0$,

$$P = \frac{ma}{g}. \tag{2}$$

If the force P acts at a distance l from the axis of rotation, its torque is Pl. By Formula XXX

$$Pl = \frac{I\alpha}{g}. \tag{3}$$

When the angular acceleration is α, the linear acceleration of the center of mass is

$$a = \bar{r}\alpha. \tag{4}$$

Substituting $P = \frac{m\bar{r}\alpha}{g}$ in Equation (3)

$$\frac{m\ \alpha l}{g} = \frac{I\alpha}{g}; \tag{5}$$

$$m\bar{r}l = I = mk^2; \tag{6}$$

$$l = \frac{k^2}{\bar{r}}. \tag{7}$$

Equation (7) states that the center of percussion is located at a distance from the axis equal to the square of the radius of gyration divided by the distance from the axis to the center of gravity. A comparison with Art. 226 shows that the center of percussion lies on the axis of oscillation.

Problems

1. A uniform rod of length L is suspended on an axis through one end perpendicular to its length. Find the center of percussion.

Ans. $\frac{2}{3}L$ from the axis of suspension.

2. Find the center of percussion of a uniform rod suspended by an axis at one-third its length from one end.

3. Find the center of percussion of a uniform rod for an axis at three-eighths the length from one end.

Ans. $\frac{1}{6}L$ from the other end.

4. A bat in the form of a frustrum of a cone is 1.5 inches in diameter at one end, 3 inches in diameter at the other end, and 3 feet long. It is gripped 4 inches from the small end. If the bat is swung around the point at which it is gripped, where should it strike the ball in order that the jar on the hand may be a minimum?

5. A triangular paddle is made of a board of uniform thickness. Where is the center of percussion when the axis of suspension passes through the vertex parallel to the face of the board, and where is the center of percussion when the axis of suspension passes through the vertex perpendicular to the face of the board?

The center of percussion may also be defined as the point about which the moment of momentum is zero. In Fig. 326, O is the axis of rotation perpendicular to the plane of the paper and C is a parallel axis through the center of percussion. The center of gravity of the body lies in the plane through the axis of rotation and C. An element of mass dm at a distance r from the axis of rotation has a linear velocity of $r\omega$. The moment of momentum of this element with respect to the axis through C is the linear momentum multiplied by the length CB drawn from C perpendicular to the direction of the linear velocity. If l is the distance from O to C and θ is the angle between r and OC,

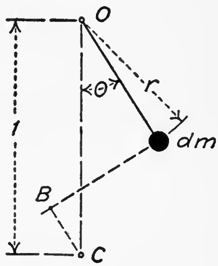

FIG. 326.

$$CB = (l - r \sec \theta) \cos \theta = l \cos \theta - r. \tag{8}$$

The moment of momentum of the element dm is

$$\text{moment of momentum} = \omega(l\,r \cos \theta - r^2)dm. \tag{9}$$

Integrating Equation (9) and equating to zero,

$$l\int r \cos \theta \, dm = \int r^2 dm. \tag{10}$$

Since $\int r \cos \theta \, dm$ is the moment of the entire mass with respect to the plane through the axis of suspension perpendicular to OC,

$$\int r \cos \theta \, dm = \bar{r}m, \tag{11}$$

in which $\bar{r}$ is the distance of the center of mass from the axis of suspension. Equation (10) then becomes

$$l\bar{r}m = k^2 m; \tag{12}$$

$$l = \frac{k^2}{\bar{r}}. \tag{7}$$

236. Summary.—The product of the mass of a body multiplied by its linear velocity is called its momentum. The momentum of a body or system of bodies is constant unless changed by the application of force from another body or system. Momentum is a vector quantity.

The product of force multiplied by time is called the impulse of the force. The impulse on a body is proportional to the change of momentum. With absolute units, the impulse is numerically equal to the change of momentum. With gravitational units, the impulse is numerically equal to the change of momentum divided by g.

Since action and reaction are equal, when two bodies exert force on each other, their impulses are equal. As a result the bodies suffer equal and opposite changes of momentum. The total change of momentum of two bodies due to forces which act between the bodies is zero.

When inelastic bodies collide, their velocity after impact is calculated from the fact that the total momentum is not changed. When elastic bodies collide, they approach for a short time. At the end of this interval, which is called the stage of compression, the bodies are moving with equal velocities in the same direction. The common velocity at the end of the compression stage is the same as it would be if the bodies were inelastic. After they reach the end of the compression stage, the elasticity of the bodies causes them to separate. The interval from the end of the compression stage until the bodies cease to touch is the stage of restitution. The ratio of the impulse during restitution to the impulse during compression is the coefficient of restitution. When the coefficient of restitution is unity, the bodies are perfectly elastic and no energy is lost in the collision.

The product of the momentum of a body multiplied by the perpendicular distance of its line of action from a given axis is the moment of momentum of the body with respect to that axis. The moment of momentum can be changed only by torque from an outside body.

The center of percussion with respect to an axis is the point at which an impulse may be given to the body in a direction normal to the plane through the axis and the center of gravity without causing any change in the reaction of the axis in the direction of the impulse. The center of percussion falls on the axis of oscillation. The moment of momentum of a rotating body about an axis through the center of percussion parallel to the axis of rotation is zero.

CHAPTER XXVI

ENERGY TRANSFER

237. Units of Energy.—In English-speaking countries, energy is measured by engineers in foot-pounds. Where the metric system is used, the engineering unit of energy is the kilogram-meter. Since 1 kilogram is equal to 2.20462 pounds, and 1 meter is equal to 3.28083 feet,

$$1 \text{ kilogram-meter} = 7.2330 \text{ foot-pounds;}$$
$$1 \text{ foot-pound} = 0.13826 \text{ kilogram-meters.}$$

For some purposes, energy is expressed in gram-centimeters.

$$1 \text{ foot-pound} = 13{,}826 \text{ gram-centimeters.}$$

For scientific purposes, work and energy are measured in ergs. An erg is the work done by the force of one dyne when the point of application moves one centimeter in the direction of the force. At the standard location, the acceleration of gravity is $32.174 \times 30.4801 = 980.67$ cm. per sec. per sec.

$$1 \text{ gram-centimeter} = 980.67 \text{ ergs.}$$
$$1 \text{ foot-pound} = 13{,}558{,}700 \text{ ergs.}$$

Physicists use the joule as a unit of work or energy.

$$1 \text{ joule} = 10{,}000{,}000 \text{ ergs.} = 10^7 \text{ ergs.}$$
$$1 \text{ foot-pound} = 1.35587 \text{ joules.}$$

Problems

1. A 40-pound mass is pulled up a 30-degree inclined plane by a force parallel to the plane. The coefficient of friction is 0.1. How much work is done and how much is the potential energy increased when the body is moved 50 feet? *Ans.* 1173.2 ft.-lb.; 1000 ft.-lb.

2. Express the answers to Problem 1 in kilogram-meters.

3. A 4-pound mass is lifted vertically a distance of 25 feet at a place where $g = 32.2$. Find the work in joules. *Ans.* 135.69 joules.

238. Power.—Rate of working is called *power*. The unit of power in English-speaking countries is the horsepower.

1 horsepower = 550 foot-pounds per second.
1 horsepower = 33,000 foot-pounds per minute.

For electrical measurement, the *watt* is the unit of power.

1 watt = 1 joule per second = 10^7 ergs per second.
1 horsepower = 550 × 1.35587 = 745.7 watts.

It is customary to use

1 horsepower = 746 watts.

For direct currents and for alternating currents in non-inductive circuits, the power in watts is equal to the product of the potential difference of the terminals of the circuit in volts multiplied by the current in amperes.

$$\text{Watts} = EI = \text{volts} \times \text{amperes}.$$

Since $E = RI$ in a non-inductive circuit,

$$\text{watts} = I^2R,$$

in which R is the resistance in ohms.

Problems

1. A tank, 20 feet square and 30 feet deep, is filled with water from a source which is 40 feet below the bottom of the tank. If the intake pipe enters the bottom of the tank, what is the horsepower required to fill it in 50 minutes. *Ans.* 25 hp.

2. The pump in Problem 1 has an efficiency of 80 per cent and is driven by a motor which has an efficiency of 90 per cent. How many kilowatts are required to drive the motor? *Ans.* 25.9 kilowatts.

3. A conical tank is 20 feet in diameter at the top and 30 feet deep. It is filled with water from a source 40 feet below the bottom by means of a pump of 75 per cent efficiency which is driven by a 220-volt motor of 90 per cent efficiency. If the tank is filled in 30 minutes through a pipe which enters at the bottom, what is the average current supplied to the motor?

4. How many horsepower equal one kilowatt?

5. A flywheel 6 feet in diameter is making 300 revolutions per minute. The tension on one side of the belt is 1800 pounds and the tension on the other side is 300 pounds. Find the horsepower transmitted. *Ans.* 25.7 hp.

6. If T is the torque in foot-pounds in a shaft, show that the work per revolution is $2\pi T$.

239. Mechanical Equivalent of Heat.—Energy can neither be created nor destroyed. When it apparently disappears, it merely changes its form. The statement that the total energy of a closed system remains unchanged is called the Law of Conservation of Energy.

The most common change of energy is the change to the form of heat energy. When friction takes place between two bodies, heat is generated. The temperature of the bodies may be so raised by the heat of friction that the bodies are melted or ignited.

Heat energy is measured in British thermal units or in calories. A British thermal unit is the heat required to raise the temperature of one pound of water one degree Fahrenheit. Since the specific heat of water varies slightly with the temperature, a British thermal unit is defined more accurately as the amount of heat required to raise the temperature of one pound of water from 32 degrees to 33 degrees Fahrenheit.

A calorie is the amount of heat required to raise the temperature of one kilogram of water one degree Centigrade. A calorie is defined accurately as the amount of heat required to raise the temperature of one kilogram of water from 0 degrees to 1 degree Centigrade.

A small calorie is the amount of heat required to raise the temperature of 1 gram of water from 0 degrees to 1 degree Centigrade. The small calorie is called a calorie by physicists.

The experiments of Joule and Rowland have shown that 427 kilogram-meters of work will raise the temperature of 1 kilogram of water 1 degree Centigrade. This figure is called the *mechanical equivalent of heat.*

$$1 \text{ large calorie} = 427 \text{ kilogram meters} = J.$$
$$1 \text{ British thermal unit} = 778 \text{ foot-pounds.}$$

Problems

1. If the work done in lifting 1 kilogram a distance of 427 meters is sufficient to raise the temperature of 1 kilogram of water 1 degree Centigrade, how high will a mass of 1 pound be lifted by the energy which is required to raise the temperature of 1 pound of water 1 degree Centigrade?

Ans. $h = 1400$ ft.

2. If the energy required to raise the temperature of 1 pound of water 1 degree Centigrade is sufficient to lift 1 pound a vertical distance of 1400 feet, how high will the energy required to raise the temperature of 1 pound of water 1 degree Fahrenheit lift 1 pound?

3. A friction brake which is absorbing 30 horsepower is cooled by water at 20 degrees Centigrade. The water is evaporated from and at 100 degrees Centigrade. How many pounds of water are required per hour?

Ans. 68.7 lb.

4. A sled weighing 1200 pounds is drawn along a horizontal ice surface by a horizontal pull of 70 pounds. If the temperature of the ice is 32 degrees Fahrenheit, how much ice is melted when the sled moves 100 feet?

5. An iron shot moving at a speed of 1200 feet per second is stopped by striking an object. If the specific heat of iron is 0.11 and if one-half of the energy goes to heat the shot, how high is its temperature raised?

6. A meteorite strikes the air with a velocity of 6 miles per second relative to the earth. If the specific heat is 0.11 and if one-half of the energy goes to heat the meteorite, how high is its temperature raised?

7. The drum of Fig. 327 is mounted on a smooth pivot. The vertical shaft at the top drives a set of paddles inside the drum. The drum is filled with water which is heated by the mechanical energy of the rotating paddles.

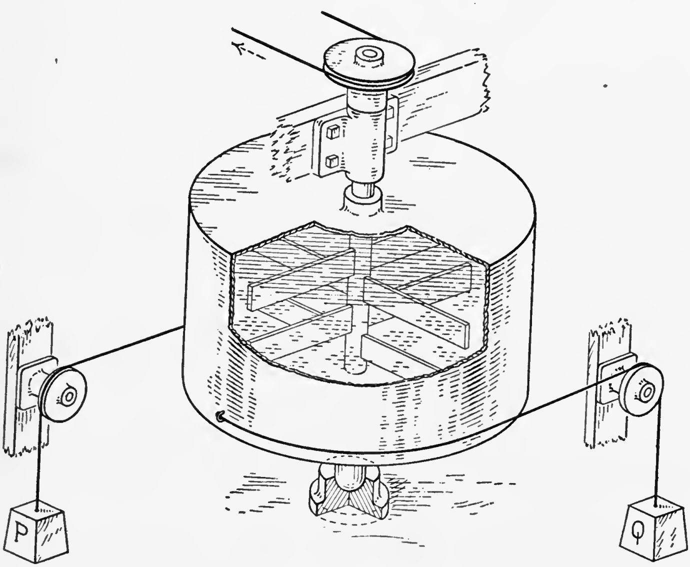

FIG. 327.

The torque is measured by means of two cords which are wound round the drum and run over smooth pulleys. The water equivalent of heat of the apparatus was found to be 2.42 pounds. With 17.58 pounds of water in the drum, the temperature was raised from 50° Fahr. to 74° Fahr. while the shaft made 4000 revolutions. The average torque during the experiment was 10.8 foot-pounds. Find the mechanical equivalent of heat.

8. How many kilowatts are required to raise the temperature of 20 pounds of water from 62 degrees to boiling in 30 minutes?

9. How many kilowatt-hours are required to raise 10 kilograms of water from 20 degrees C. to 100 degrees C.?

10. Coal with a calorific value of 13000 B.t.u per pound is used with a boiler of 70 per cent efficiency and an engine of 15 per cent efficiency. How many pounds of coal are required per horsepower-hour? *Ans.* 1.87 lb.

240. Power of a Jet of Water.—When a liquid flows from an orifice, the velocity is the same as that which a body would acquire in falling freely from a height equal to the vertical distance of of the orifice below the surface the liquid. In Fig. 328,

$$v = \sqrt{2gh}, \qquad (1)$$

in which h is measured from the center of the orifice and v is the velocity at the center. If h is measured from any other point in the orifice, the velocity v is the velocity at that point. For an

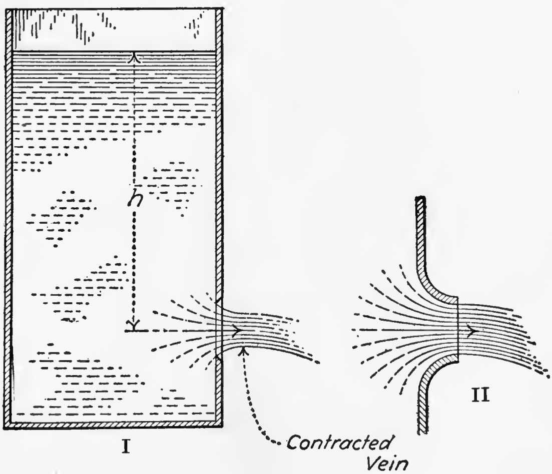

FIG. 328.

orifice of small dimensions compared with the height of the water, the velocity at the middle agrees with the average velocity within narrow limits.

The velocity of Equation (1) is called the theoretical velocity. Fig. 328, I, shows an orifice in a thin plate (sometimes called a frictionless orifice). Practically no energy is lost in an orifice of this kind, and Equation (1) gives the true velocity. When an orifice of this kind is placed in a plane surface of dimensions considerably greater than the diameter of the orifice, the momentum of the liquid flowing in from all sides tends to carry it in past the edge. As a result of this transverse component, the motion at the surface is not normal to the plane of the orifice and the jet is contracted on all sides. At a distance from the orifice about equal to its radius, all the liquid is moving in one direction and the diameter of the jet is considerably smaller than that of the orifice. This section of the jet is called the *contracted vein.* The ratio of the area of the contracted vein to the area of the orifice is called the *coefficient* of *contraction.* With full contraction, the coefficient of contraction for circular orifices is about 0.62.

The quantity of liquid which flows from an orifice in one unit of time is equal to the area of the contracted vein multiplied by the average velocity. The quantity is given by the equation,

$$Q = KA\sqrt{2gh}, \tag{2}$$

in which A is the area of the orifice, and K is a coefficient called the *coefficient of discharge.* The coefficient of discharge is the ratio of the quantity actually discharged to the quantity which would flow through the orifice if there were no contraction and the actual velocity were the full theoretical velocity. In an orifice in a thin plate, the velocity is the full theoretical velocity and the coefficient of discharge is equal to the coefficient of contraction. The coefficient of contraction for an orifice of this kind is 0.62. The equation

$$Q = 0.62A\sqrt{2gh}$$

gives the discharge from an orifice in a thin plate.

Figure 328, II, shows a gradually converging short nozzle. Since the area of the jet is equal to the area of the nozzle, the coefficient of contraction is unity. The friction slightly reduces the velocity at the surface; the coefficient of discharge is, therefore, a little less than unity. With the proper curvature, K is greater than 0.99. With ordinary nozzles, K is about 0.95.

The power of a jet may be found by means of the kinetic energy per pound of liquid. Since the energy is equivalent to the work done in lifting a pound a vertical distance of h feet, the power is easiest calculated by multiplying the number of pounds per second by the height h.

Example

Water flows from a 6-inch orifice in a thin plate under a head of 60 feet. The coefficient of contraction is 0.62. Find the horsepower of the jet.

$$v = \sqrt{2 \times 32.174 \times 60} = 62.14 \text{ ft. per sec.}$$

$$Q = 0.62 \times \frac{\pi}{16} \times 62.14 = 7.582 \text{ cu. ft. per sec.}$$

Since one cubic foot of water weighs nearly 62.5 pounds, the energy of each cubic foot is $62.5 \times 60 = 3750$ ft.-lb. The total energy is $7.582 \times 3750 = 28{,}432$ ft.-lb. per second. This is 51.7 horsepower.

Pressure of a liquid or gas is often given in pounds per square inch. A column of water 1 inch square and 1 foot high weighs $\frac{62.5}{144} = 0.434$ pound. A column 1 inch square and 2.30 feet high weighs 1 pound. To change water pressure in pounds per square inch to head in feet of water, multiply by 2.30.

Problems

1. Water flows from a 4-inch circular orifice in a thin plate under a pressure of 40 pounds per square inch. Find the horsepower of the jet.

Ans. 43.52 hp.

2. The jet from the orifice of Problem 1 drives a water wheel of 80 per cent efficiency which drives a generator of 90 per cent efficiency. How many kilowatts are generated? *Ans.* 23.38 kw.

3. If the jet from a given orifice delivers 60 hp. when the head is 40 feet of water, what will it deliver when the pressure is 50 pounds per square inch?

241. Work of an Engine.—The work done by the steam in an engine during one stroke is the product of the total pressure on one side of the piston multiplied by the length of the stroke.

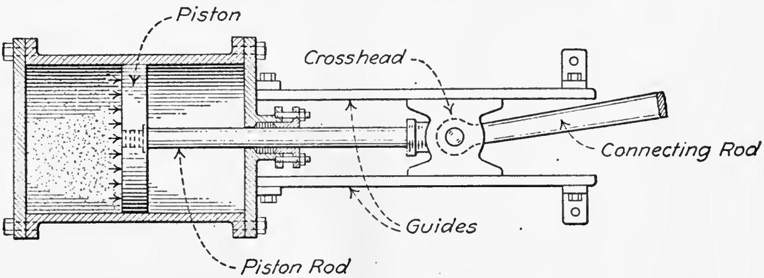

FIG. 329.

The total pressure is the pressure per square inch multiplied by the area of the piston. If the diameter of the cylinder is 20 inches, the length of the stroke is 48 inches, and the average pressure on one side of the piston during the stroke is 50 pounds per square inch, the total work of the steam on that side is

$$314.16 \times 50 \times 4 = 62{,}832 \text{ foot-pounds.}$$

If the pressure of 50 pounds per square inch is measured from the atmospheric pressure as the zero (as is customary) and if the opposite end of the cylinder is entirely open during the entire stroke, then 62,832 foot-pounds represents the actual work done on the piston. Usually, there is some back pressure in the opposite end of the cylinder; the effective work on the piston is, therefore, somewhat less than the work of the steam on one side. The actual work is the difference between the positive work in the steam end of the cylinder and the negative work in the other end.

The average pressure in an engine cylinder is determined by means of an indicator, Fig. 330, I. An indicator is a pressure gage which is connected to the cylinder by means of a short steam pipe. An indicator consists essentially of a small cylinder in which moves a light piston. The steam pressure in the cylinder of the gage is balanced by a calibrated spring on the opposite side of the piston. The piston actuates a light lever, which carries a pencil at the end. When a sheet of paper is

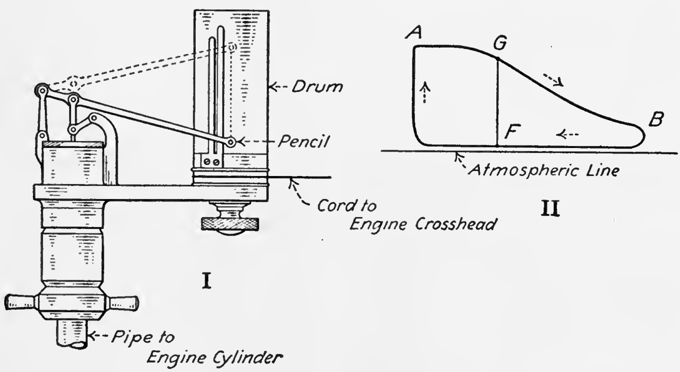

Fig. 330.

moved perpendicular to the motion of the pencil, a curve whose ordinates give the pressure at every instant is drawn. The paper is placed on a drum which is attached to the engine in such way that its motion is proportional to the motion of the piston.

Figure 330, II, shows an indicator card. The arrows give the direction of the motion of the pencil relative to the paper. The distance of any point, such as G, above the atmospheric line measures the pressure in the cylinder of the engine at the corresponding point of its stroke. The effective force on the piston is the difference between this pressure and the back pressure on the opposite side of the piston. This back pressure may be obtained by a second indicator attached to the opposite end of the cylinder. It is not necessary to consider together the pressure on one side of the piston and the back pressure on the other side in order to calculate the power. The effective work during one cycle may be calculated from the difference of pressure on the *same* side of the cylinder. In Fig. 330, II, the height *FG* of the indicator card gives the difference in pressure during the forward and backward strokes. The average net height of the indicator card, as obtained by dividing its area by its length, gives the mean effective pressure for the one end of the cylinder.

Problems

1. The indicator card from the head end of an engine cylinder has an area of 8.40 square inches, and the card from the crank end has an area of 8.64 square inches. Each card is 8 inches long. The indicator spring used in this test is compressed 1 inch by a pressure of 40 pounds per square inch in the cylinder of the indicator. Find the mean effective pressure.

Ans. m.e.p. = 42 pounds per square inch in the head end, and 43.2 pounds per square inch in the crank end.

2. The cylinder of the engine in Problem 1 is 20 inches in diameter and the piston rod is 4 inches in diameter. The stroke is 48 inches and the speed is 80 revolutions per minute. Find the indicated horse power.

Ans. 284.5 hp.

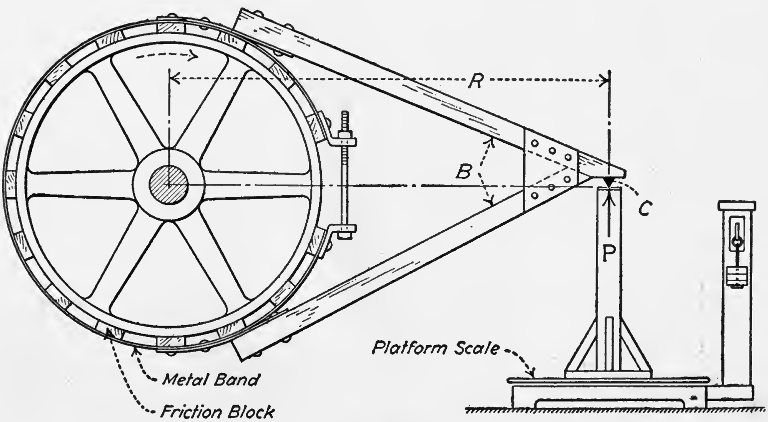

FIG. 331.

242. Friction Brake.—Fig. 331 shows one type of friction brake, such as is used to *absorb* and *measure* the power developed by an engine or motor. The form shown consists of a number of wooden friction blocks placed around the rim of the pulley. The blocks are fastened to a flexible metal band. The ends of the band are connected together by bolts. By turning the nuts on these bolts the tension may be varied and the pressure of the blocks on the pulley may be adjusted to give any desired value to the friction. The arm *B* ends with a knife-edge *C* at a distance *R* from the axis of the pulley. The knife-edge rests on a vertical support, which stands on a platform scale. When the pulley is turning in the direction of the arrow, the friction of the blocks transmits a torque to the brake. This torque is balanced by the upward force P from the support to the knife-edge.

Suppose that the pulley is held stationary and that a force P is applied to the knife-edge at right angles to the radius R. If the torque of this force is sufficient to balance the torque of friction, the knife-edge will describe a circle of radius R about the axis of the pulley. The circumference of this circle is $2\pi R$, and the work done by the force P during one revolution is

$$U = 2\pi RP = 2\pi T, \tag{1}$$

in which T is the torque.

If the brake is held stationary by the reaction of the support and the pulley is rotated, the work per revolution is the same as it would be if the pulley were stationary and the brake were rotated. In all cases, the work per revolution = $2\pi T$.

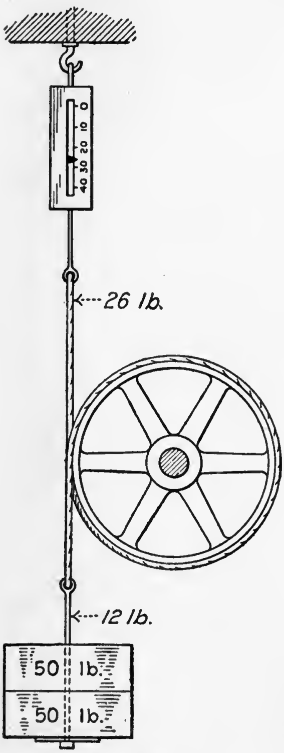

Fig. 332.

Problems

1. The flywheel of an engine makes 300 revolutions per minute. A friction brake on the wheel has a moment arm of 90 inches. Find the horsepower when the reaction at the end of the brake arm is 420 pounds.

Ans. 178 hp.

2. The arm of a brake which is attached to the pulley of a motor is 30 inches long. The motor makes 1200 revolutions per minute. Find the horsepower when the reaction at the support is 60 pounds. If the current used to drive the motor is 130 amperes at 220 volts, what is the efficiency?

Ans. 34.27 hp.; 89.3 per-cent.

3. The pulley of Fig. 332 is 10 inches in diameter and the rope is ¾ inch in diameter. The total weight is 112 pounds and the spring balance reads 26 pounds. Find the horsepower when the pulley is making 800 revolutions per minute. *Ans.* 5.86 hp.

4. What must be the coefficient of friction between the rope and the pulley of Problem 3?

5. The flywheel of Problem 1 is cooled by water which is held against the rim of the wheel by centrifugal force. The water is supplied to the wheel at 60 degrees Fahrenheit and is evaporated at 210 degrees Fahrenheit. How many pounds of water are required per hour?

243. Cradle Dynamometer.—The output of a small motor may be measured by means of a cradle dynamometer. The motor is mounted on a frame which is supported by a pair of knife-edges. These knife-edges are in line with the axis of the shaft of the motor. The torque of the belt on the motor tends to rotate the frame about the knife-edges. Since the knife-edges are in line with the axis of the pulley, the direct pull of the belt has no effect on the equilibrium. The torque of the belt is balanced by the weights P and the poise Q on a horizontal arm attached to the frame. When the motor is revolving clockwise with the belts extending toward the left, as shown in Fig. 333, the tension is

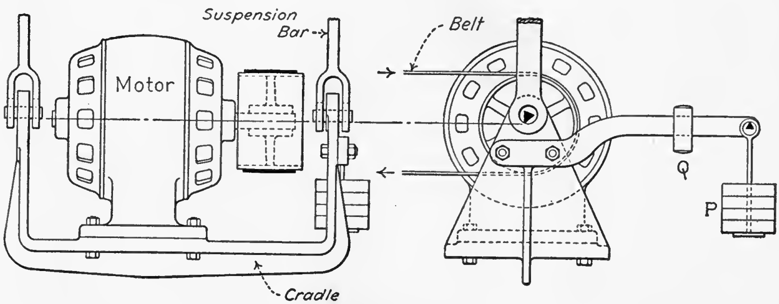

FIG. 333

greater in the upper belt and the torque from the belt to the armature is counter-clockwise. This torque is transmitted electromagnetically from the armature to the field, and then transmitted mechanically from the field to the dynamometer.

When a dynamo which is driven by a belt is to be tested, either the rotation must be opposite that shown in Fig. 333, or the arm which carries the balancing weights must extend toward the left.

The calculation of power of a cradle dynamometer is the same as the calculation for a friction brake. The work per revolution $= 2\pi T$.

244. Transmission Dynamometer.—A transmission dynamometer transmits and measures power without dissipating it by transfer into heat. Fig. 334 shows one type of a dynamometer driven by a belt. The pulley B is driven from pulley A through the dynamometer pulleys C and D. The dynamometer is arranged to measure the pull Q in the part of the belt running from C to B and the pull P in the part of the belt running from B to D. The difference $P - Q$ multiplied by the radius of the pulley B (measured to the center of the belt) gives the torque. Each dynamometer pulley is mounted on a frame which is supported by knife-edges. The knife-edges which support pulley C are in the plane of the center of the belt running from A to C and the knife-edges which support pulley D are in the plane of the center of

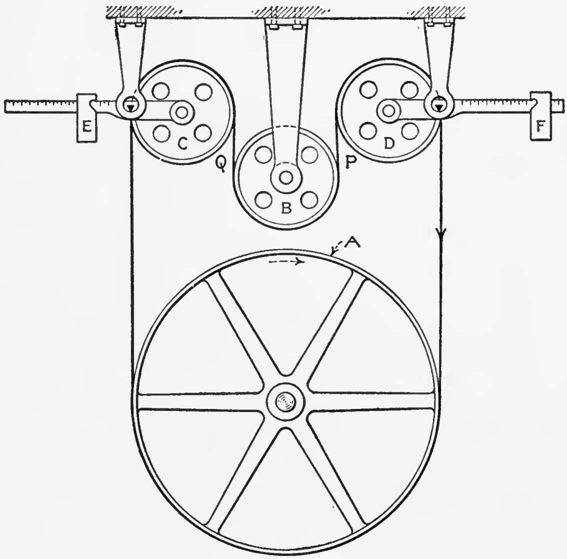

FIG. 334.

the belt running from D to A. The tension in these parts of the belt, therefore, exert no torque on the frames which support the dynamometer pulleys. The moment on the left frame is the product of the pull Q multiplied by the diameter of pulley C (from center to center of belt): and the moment on the right frame is the product of P multiplied by the diameter of pulley D.

Figure 335 shows both frames attached to a single lever, at equal distances on opposite sides of the fulcrum G. The moment on this lever measures the difference $Q - P$.

It is not absolutely necessary that the belt should run vertical from A to C and from D to A. The belt may run at any angle, provided the plane of the center passes through the line of the knife-edges on each side.

Figure 336 shows the essentials of the Robinson dynamometer.

The spur gear A drives the spur gear C through the gear B. The gearwheel B is mounted on a frame which is pivoted at O in the

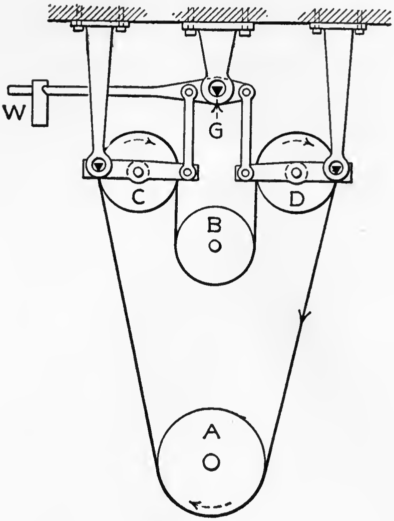

FIG. 335.

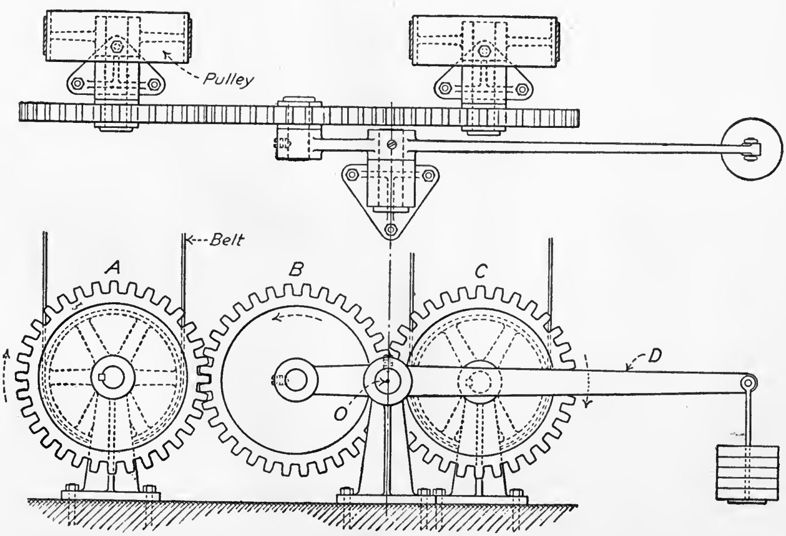

FIG. 336.

plane tangent to the pitch circle of B. The reaction between B and C does not, therefore, exert any moment on the frame.

The reaction of A on B has a moment arm equal to the diameter of the pitch circle of B. This moment is balanced by the weights at the end of the lever arm. The gearwheels A and C are mounted on a frame. Each is generally connected to a pulley by which the power is transferred to a belt.

The torque in a shaft may be measured by means of the angle of twist of a portion of its length. The angle of twist in a shaft of large diameter is very small. Delicate devices have been made, however, by means of which these small angles may be measured in a rotating shaft.

Problem

Devise a transmission dynamometer made with three bevel-gearwheels.

245. Power Transmission by Impact.—When a perfectly elastic body collides with another perfectly elastic body, there is no loss of energy. If the final velocity of one of the bodies is zero, all of its kinetic energy is transferred to the other body.

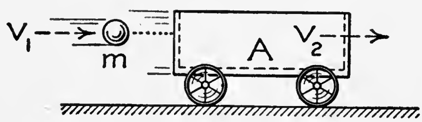

Fig. 337.

In Figure 337, A is a relatively large body which is moving with a velocity v_2, when it is struck by a relatively small body of mass m moving in the same direction with a velocity v_1. The change of velocity of m during compression is $v_1 - v_2$. If the bodies are perfectly elastic, the change during compression and restitution is $2(v_1 - v_2)$. If t is the time of contact, the average force is

$$P = \frac{2m(v_1 - v_2)}{gt} \tag{1}$$

The distance which the mass A moves during the time t is v_2t; the work done by m on A is, therefore,

$$U = \frac{2m(v_1 - v_2)v_2}{g}. \tag{2}$$

The kinetic energy of the mass m before collision was

$$U = \frac{mv_1^2}{2g}. \tag{3}$$

If all this energy is transferred to A,

$$\frac{v_1^2}{2g} = \frac{2(v_1 - v_2)v_2}{g}; \quad (4)$$

$$4v_2^2 - 4v_2v_1 + v_1^2 = 0; \quad (5)$$

$$v_2 = \frac{v_1}{2}. \quad (6)$$

When the mass of A is so large relatively that its velocity is not appreciably changed by collision with m, the velocity of A must be one-half as great as the initial velocity of m in order that all the energy of m may be transferred. The change of velocity

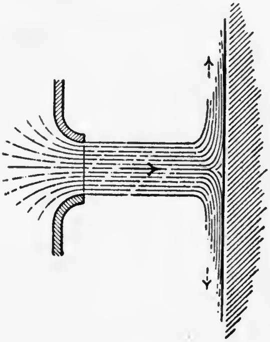

FIG. 338.

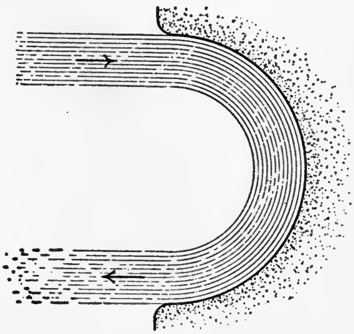

FIG. 339.

of m during compression is $\frac{v_1}{2}$ and the change during restitution is the same; the final velocity of m is zero and its kinetic energy is zero.

When a water jet strikes a plane surface, as in Fig. 338, it behaves almost as an inelastic body. If it strikes a curved surface tangentially, it is deflected with little loss of energy. If the surface is such as to deflect the jet through 180 degrees, as in Fig. 339, it behaves as a stream of perfectly elastic particles.

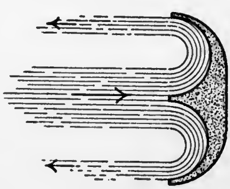

FIG. 340.

Figure 340 shows a section of one bucket of a Pelton impulse wheel, which makes use of this principle. The water jet strikes the middle of the bucket and divides. Each portion of the stream has its direction changed nearly 180 degrees. A series of such buckets is arranged around the rim of the wheel in such a way that the jet is always striking one or two buckets. The maximum efficiency is secured when the velocity of the buckets is one-half the velocity of the water in the jet.

Problems

1. A Pelton wheel is 5 feet in diameter from center to center of buckets. It is driven by a jet which has a velocity of 120 feet per second. What should be the speed of the wheel for maximum efficiency? *Ans.* 229 r.p.m.

2. What should be the linear velocity of the buckets of an impulse wheel which is driven by water under a pressure of 150 pounds per square inch?

246. Summary.—Energy is measured in foot-pounds, kilogram-meters, gramcentimeters, ergs, and joules.

Power is rate of working. A horsepower is 550 foot-pounds per second. A watt is the electrical measure of power.

$$746 \text{ watts} = 1 \text{ horsepower.}$$

The relation between heat energy and mechanical energy is called the mechanical equivalent of heat.

$$778 \text{ foot-pounds} = 1 \text{ British thermal unit.}$$

$$427 \text{ kilogram-meters} = 1 \text{ large calorie.}$$

The velocity of a liquid flowing from an orifice is that of a body which falls freely from a height equal to the depth of liquid above the orifice.

$$v = \sqrt{2gh}.$$

The quantity is the velocity multiplied by the area of the jet. For an orifice in a thin plate,

$$Q = 0.62A\sqrt{2gh}.$$

The work of a water jet per unit of time is the product of the mass per unit of time multiplied by the height of the water above the orifice.

The work of steam in a cylinder is the product of the pressure multiplied by the length of stroke. The pressure is the area of the piston multiplied by the mean effective pressure. The mean effective pressure is determined from the average height of an indicator card.

The power of a motor may be measured by means of a friction brake. The work per revolution is

$$U = 2\pi T,$$

in which T is the torque about the axis of rotation. The torque of small motors may be measured by means of a cradle dynamometer.

Transmission dynamometers measure the power of a machine without absorbing it.

When perfectly elastic bodies collide, there is no loss of energy. In order that a body may give up all of its energy, its final velocity must be zero. When a continuous stream of particles strikes a moving body, the condition of maximum efficiency requires that the body shall be moving with one-half the velocity of the particles. A jet of water which strikes a curved surface tangentially and is deflected 180 degrees behaves like a body which is highly elastic.

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